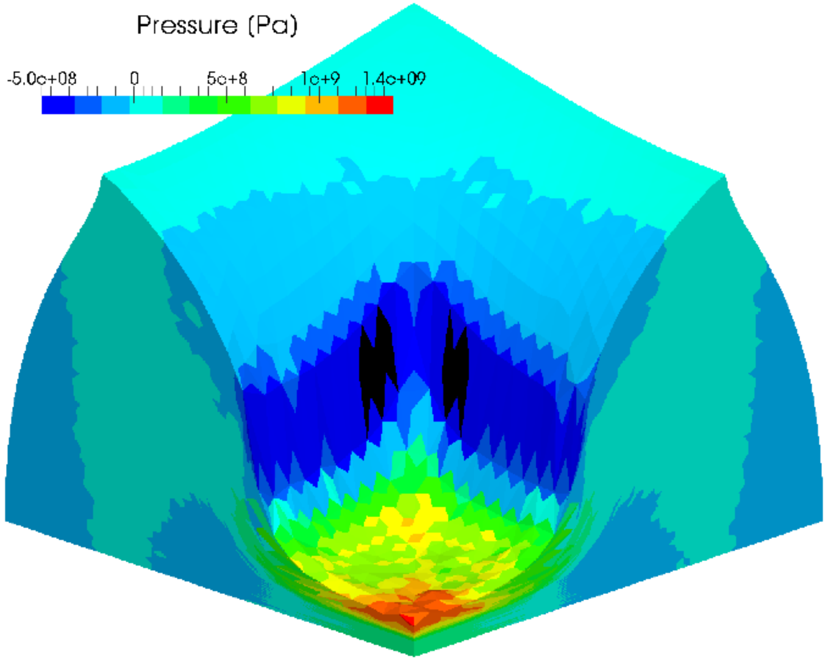
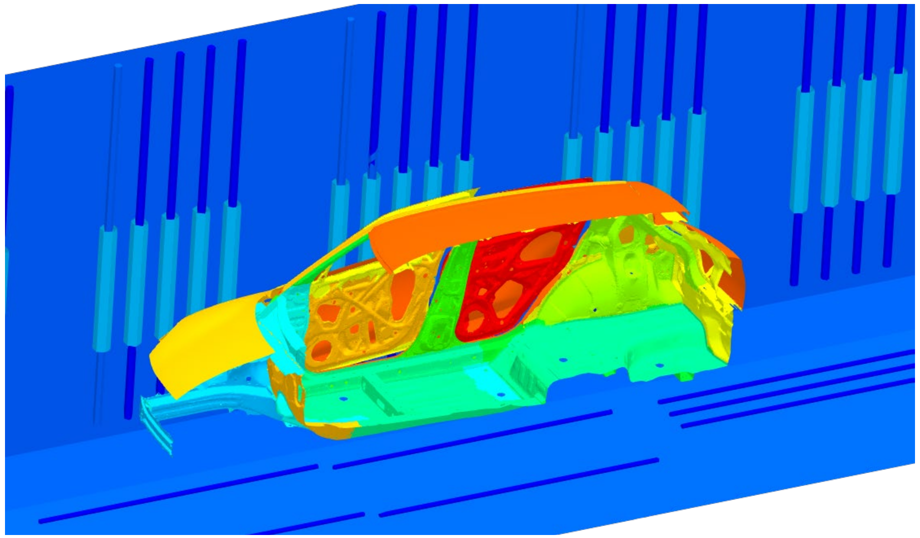
EC-SSE-SRI-T4: a next-gen smoothed finite element method for nearly incompressible large deformation analysis





Yuki ONISHI (Tokyo Institute of Technology)

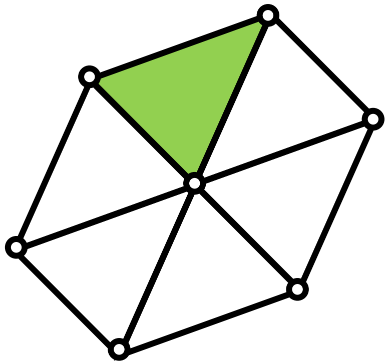
Quick Review of
Smoothed Finite Element Method (S-FEM)

What is S-FEM?

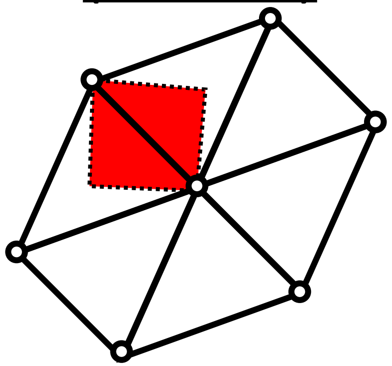
- ***Smoothed*** finite element method (**S-FEM**) is a relatively new FE formulation proposed in 2006.
- S-FEM is one of the **gradient (strain) smoothing** techniques.
- There are **many kinds of S-FEMs** depending on the scheme of smoothing.
- There are a few *classical* S-FEMs depending on the smoothing domain.

For example, in a 2D triangular mesh:

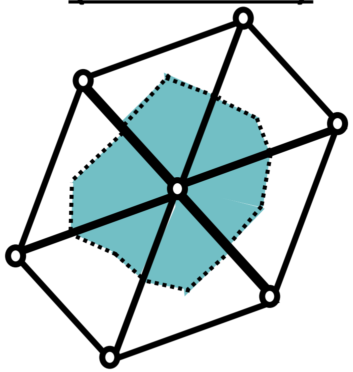
Standard FEM



Edge-based S-FEM (ES-FEM)



Node-based S-FEM (NS-FEM)



Each colored area shows the domain for gradient smoothing.

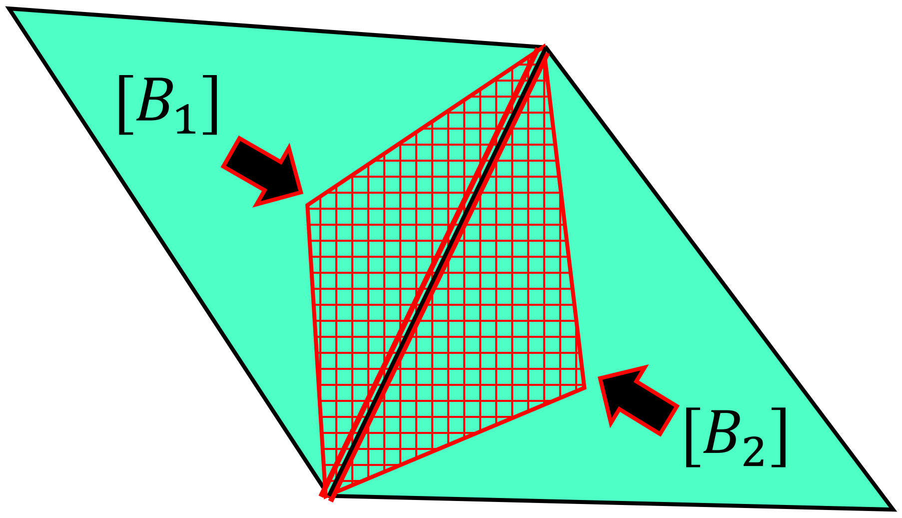
e.g.) Brief of ES-FEM

Let us consider a mesh with only two 3-node triangular cells.

Let me explain in 2D for simplicity

- Calculate $[B](= \mathrm{d}\boldsymbol{N}/\mathrm{d}\boldsymbol{x})$ at each cell as usual.
- Distribute each $[B]$ to the connecting edge with an area weight and build $[{}^{\mathrm{Edge}}B]$.
- Calculate strain ($\boldsymbol{\varepsilon}$), Cauchy stress ($\boldsymbol{\sigma}$) and nodal internal force $\{f^{\mathrm{int}}\}$ in each edge smoothing domain with $[{}^{\mathrm{Edge}}B]$.

As if putting a Gauss point on each edge center



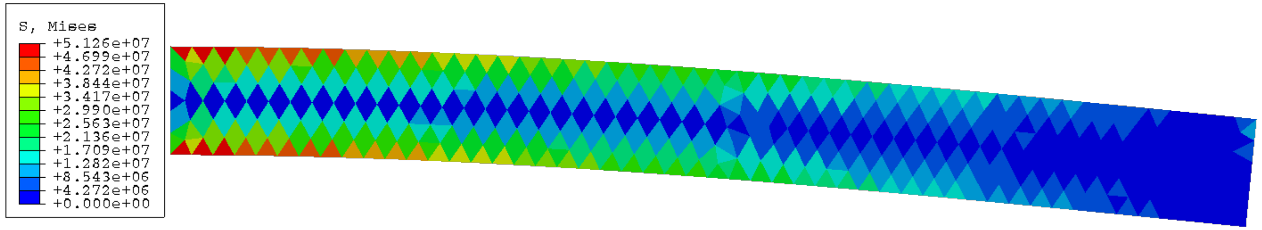
$[{}^{\mathrm{Edge}}B]$

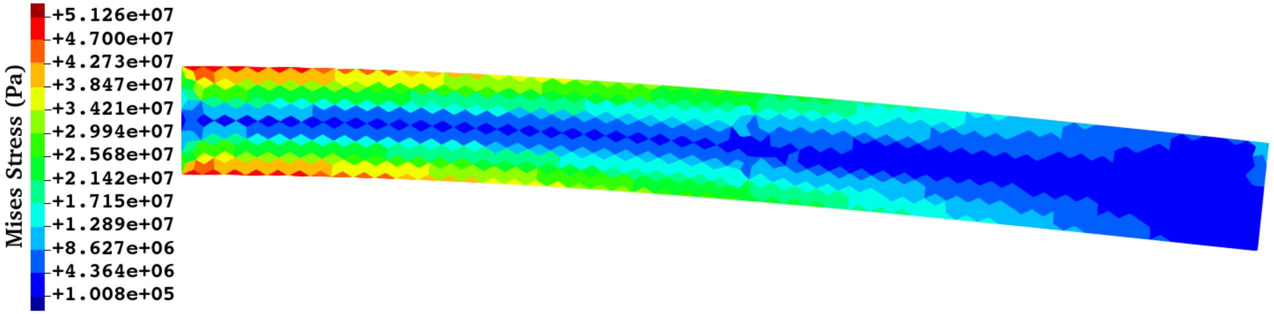
↳ ${}^{\mathrm{Edge}}\boldsymbol{\varepsilon}$, ${}^{\mathrm{Edge}}\boldsymbol{\sigma}$, $\{{}^{\mathrm{Edge}}f^{\mathrm{int}}\}$ etc.

Performance of ES-FEM

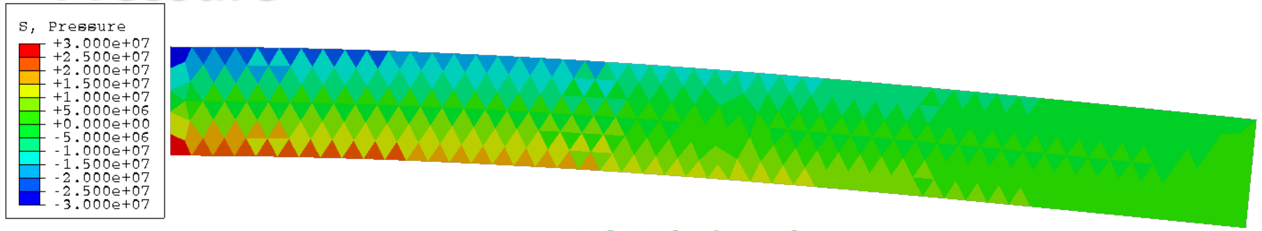
Cantilever Bending Analysis with Dead Load at the Tip Size:10x1 m, $\nu = 0.3$

Mises stress

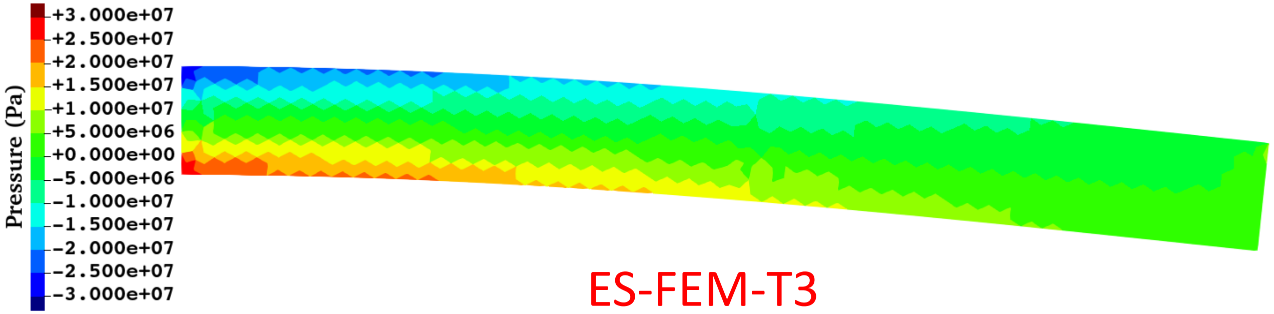




Pressure



ABAQUS CPE3
(Standard FEM-T3)

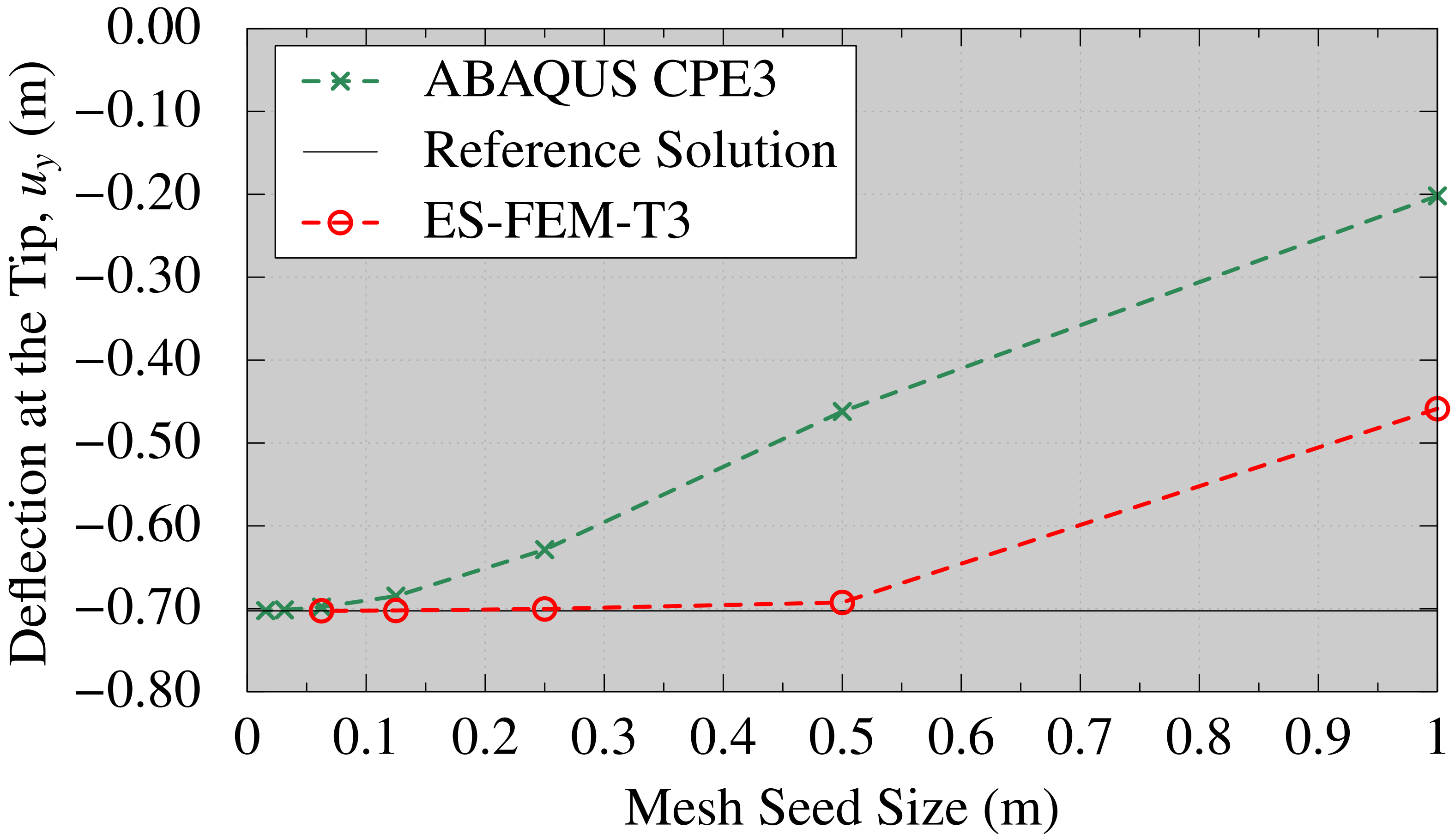


ES-FEM-T3

ES-FEM-T3 gives more accurate stress/strain distributions
than the standard FEM-T3 using the same mesh.

Performance of ES-FEM

Cantilever Bending Analysis with Dead Load at the Tip Size:10x1 m, $\nu = 0.3$



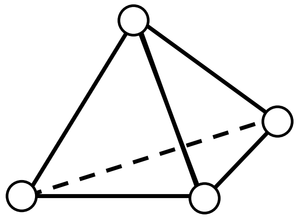
Mesh Seed Size (m)	The Number of Cells	
1	22	
1/2	84	Converged
1/4	342	
1/8	1,442	
1/16	5,766	Converged
1/32	23,034	
1/64	92,146	

ES-FEM-T3 is shear locking free and thus the mesh convergence rate of displacement/force is much faster than the Standard FEM-T3.

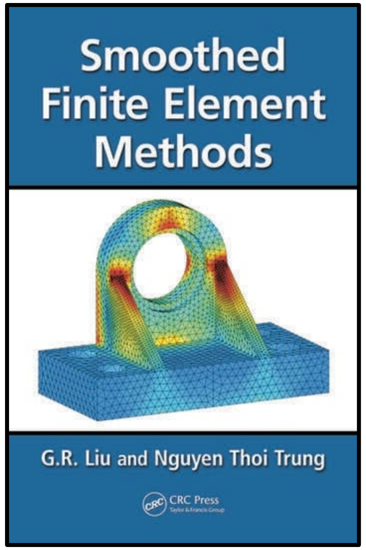
What are the benefits of S-FEM?

1. **Super-linear mesh convergence rate with T4 mesh.**
   (Almost same rate as 2nd-order elements with T4 mesh.)
2. **Shear locking free with ES-FEM-T4.**
   (Good accuracy with T4 mesh in solid mechanics.)
3. **Volumetric locking free with NS-FEM-T4.**
   (Key technique for rubber-like nearly incompressible solid.)
4. **Little accuracy loss with skewed meshes.**
   (No problem with complex geometry or severe deformation.)
5. **No increase in DOF.**
   (Purely displacement-based formulation.)
6. **Easy to code.**
   (keeping away from mixed variational formulations.)

T4: 4-node Tetrahedra



T4



G.R. Liu *et al.*, CRC Press

Search | S-FEM wiki

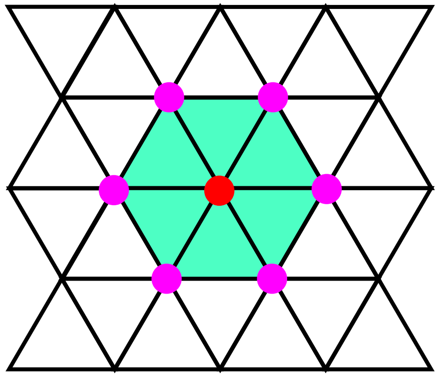
S-FEM is a powerful method suitable for practical industrial applications.

What are the drawbacks of S-FEM?

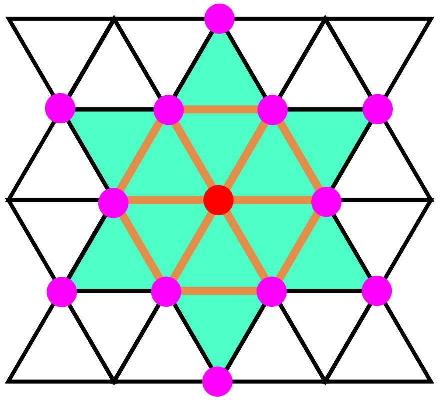
1. Wider bandwidth of stiffness matrix $[K]$.

e.g., in case of T3 mesh in 2D,

A node is
referred by 6 calls.
⇒ connecting with
7 nodes including itself.



FEM-T3 (Bandwidth: 7)



ES-FEM-T3 (Bandwidth: 13)

A node is
referred by 12 edges
that referred by 12 cells.
⇒ connecting with
13 nodes including itself.

∴ about x2 bandwidth.

(In case of T4 mesh in 3D, the bandwidth of $[K]$ is about x3 wider.)

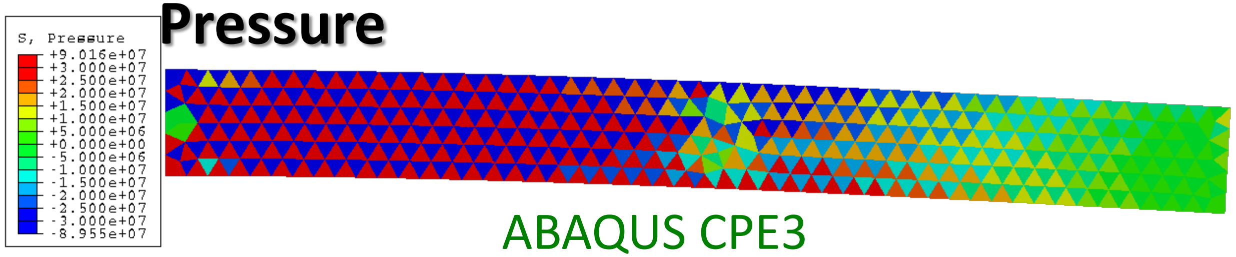
However, this drawback is NOT a major issue
because ES-FEM has **much faster mesh convergence rate** than FEM-T4.

What are the drawbacks of S-FEM?

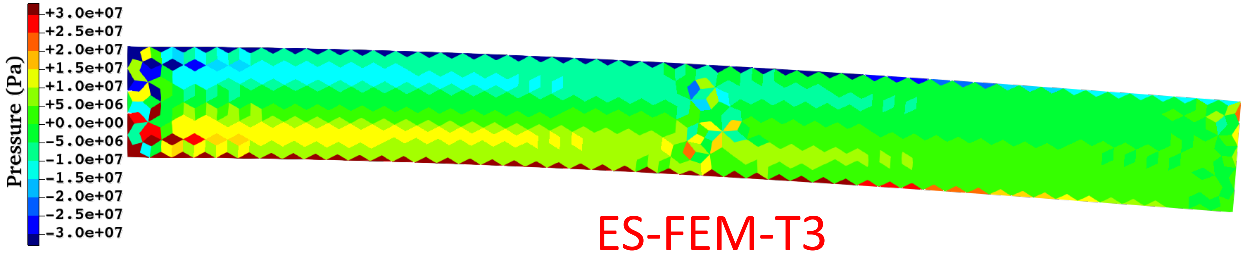
2. No low-cost formulation to suppress pressure checkerboarding.

e.g., in case of T3 mesh in 2D cantilever bending analysis with $\nu = 0.49$

Pressure



ABAQUS CPE3
(Standard FEM-T3)



ES-FEM-T3

ES-FEM shows a kind of **pressure checkerboarding** in common with the standard FEM.

3. Hard to implement to the standard FE code.

∵ S-FEM requires stain smoothing across cells.

Therefore, S-FEM formulation cannot be described as an independent element.

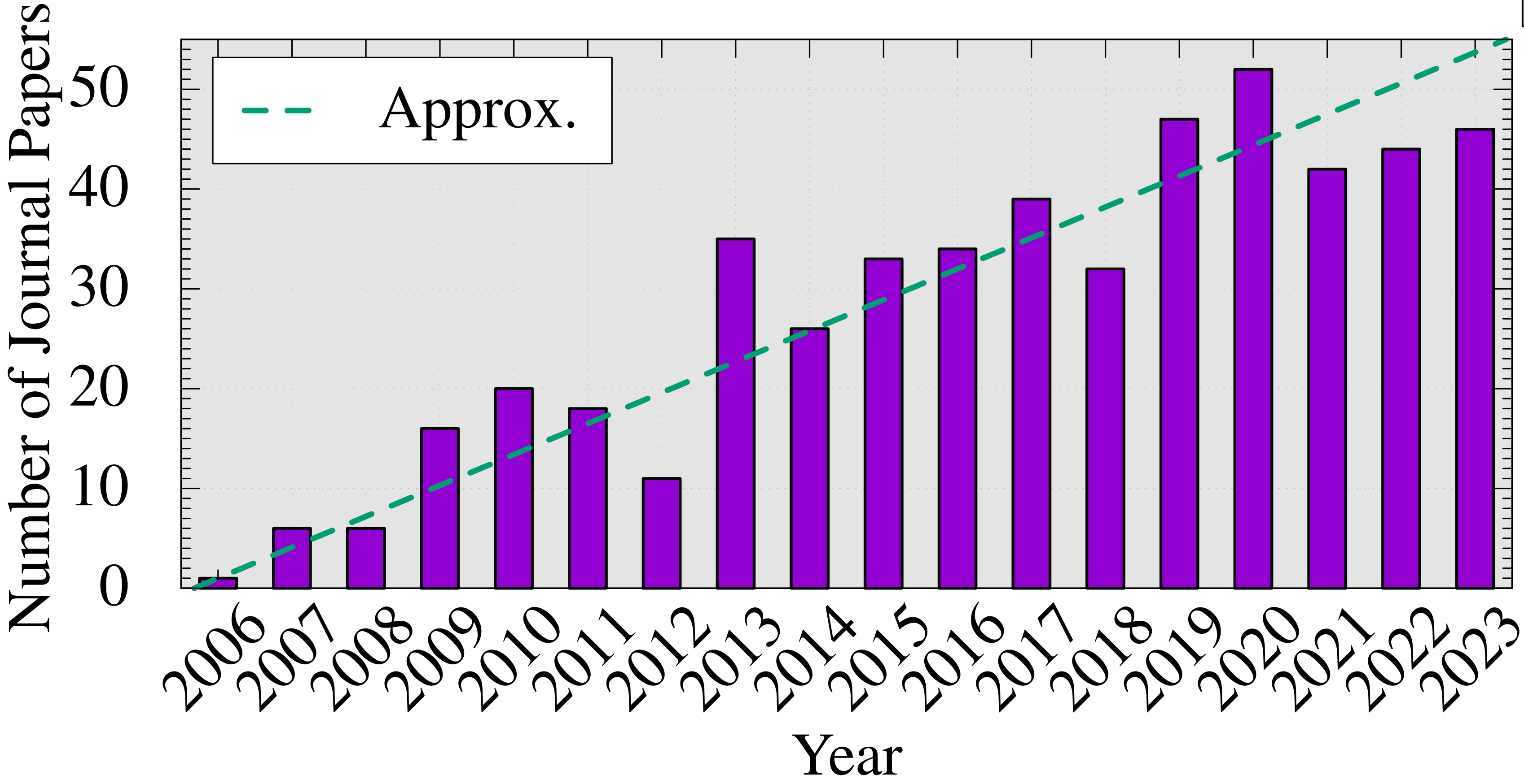
But it is OK as long as we develop our own in-house code.

A brand-new code with a novel S-FEM formulation
for nearly incompressible solids is still awaited.

How popular is S-FEM?

Number of journal papers whose **title** contains "**smoothed finite element**":

inquired at Google Scholar

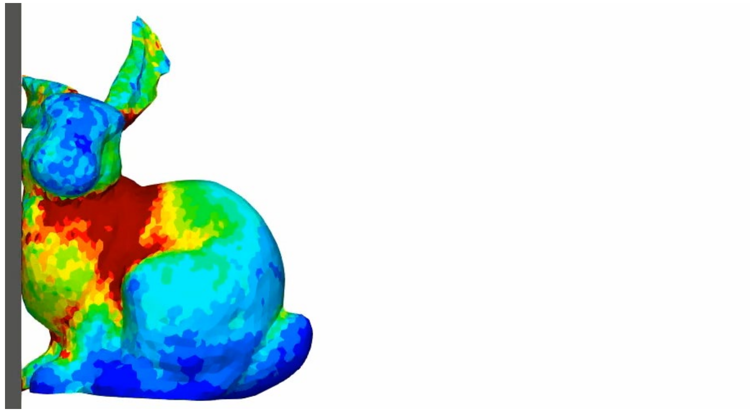


The attraction to S-FEM is expanding continuously.

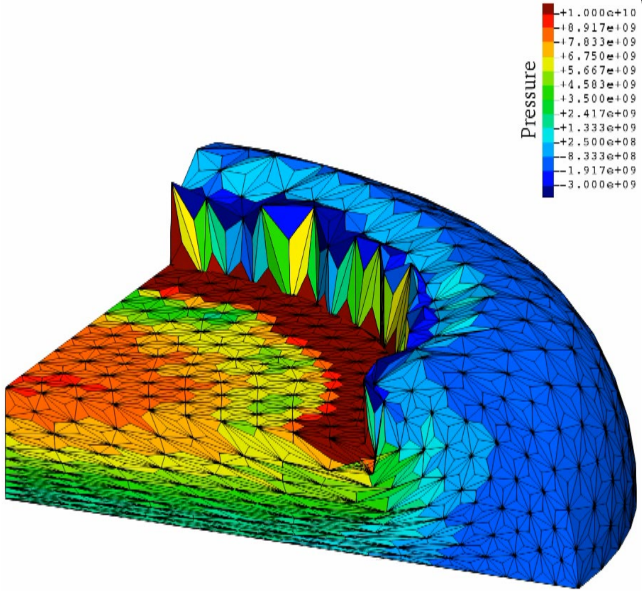
Applications of S-FEM-T4 in Our Lab

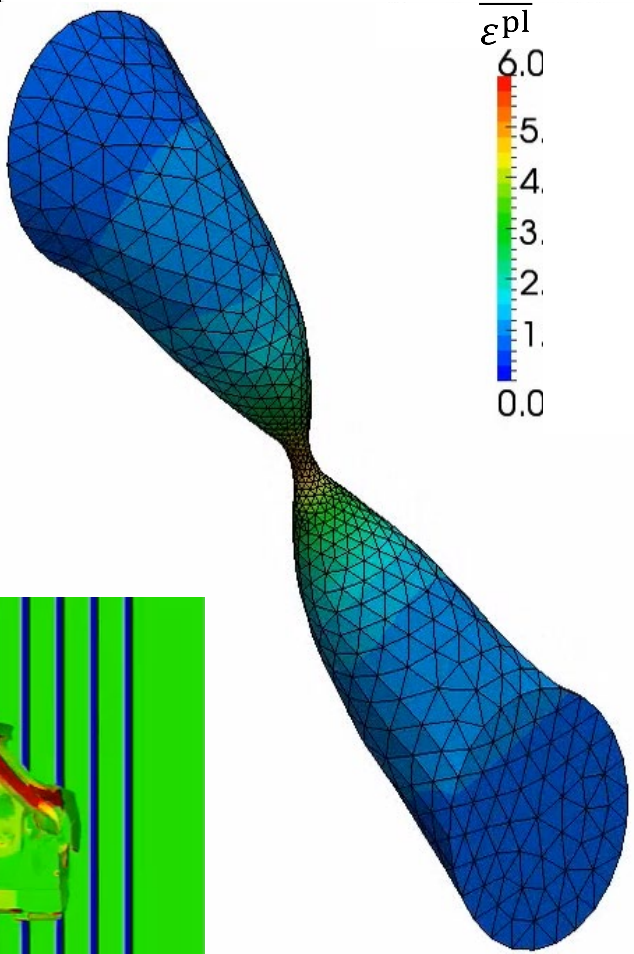
■ Large deformation solid mechanics (still in academic research)

Dynamic Explicit

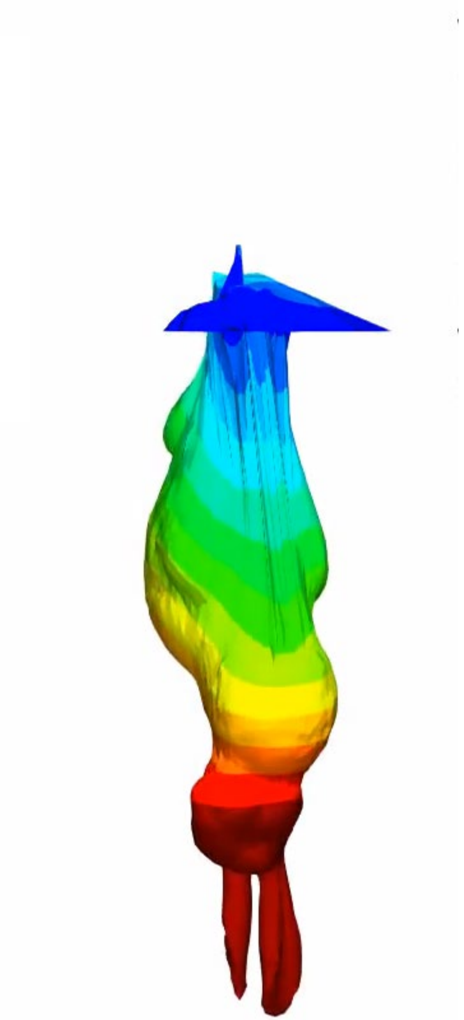


Static Implicit



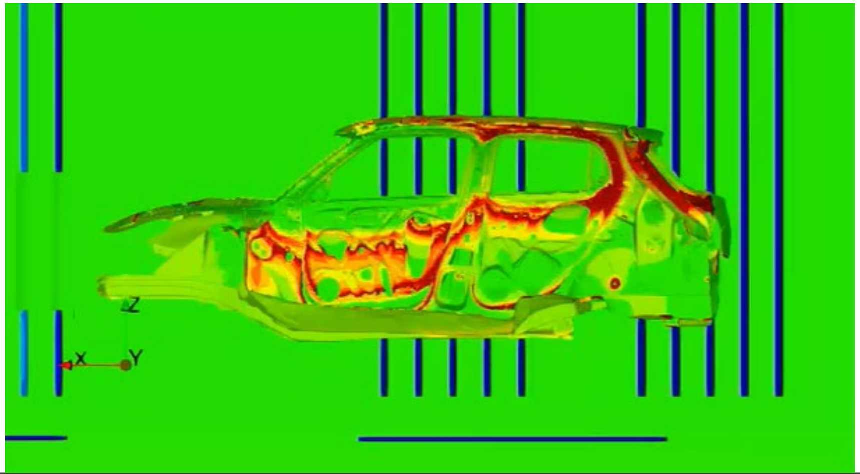


Viscous Implicit



■ Electrostatic (already in commercial use)

Carbody Electro Deposition



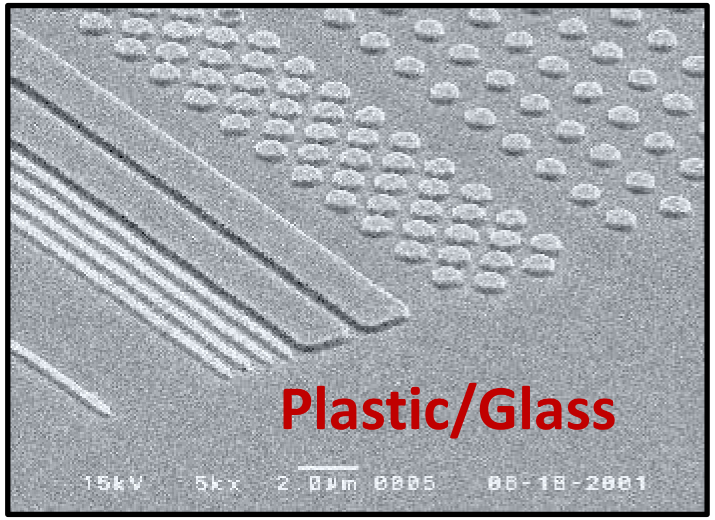
Today, I'll focus on **static implicit** analyses and introduce the latest studies on S-FEM-T4 briefly.

Motivation & Objective of Our Latest Study

Motivation

What we want to do:

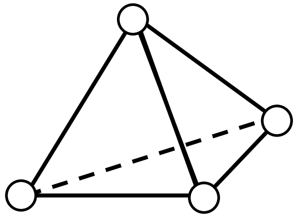
- Solve **severe large deformation** analyses accurately and robustly.
- Treat complex geometries with **tetrahedral meshes**.
- Consider **nearly incompressible materials** $(\nu \simeq 0.5)$.
- Support **contact** problems.
- Handle **auto re-meshing**.



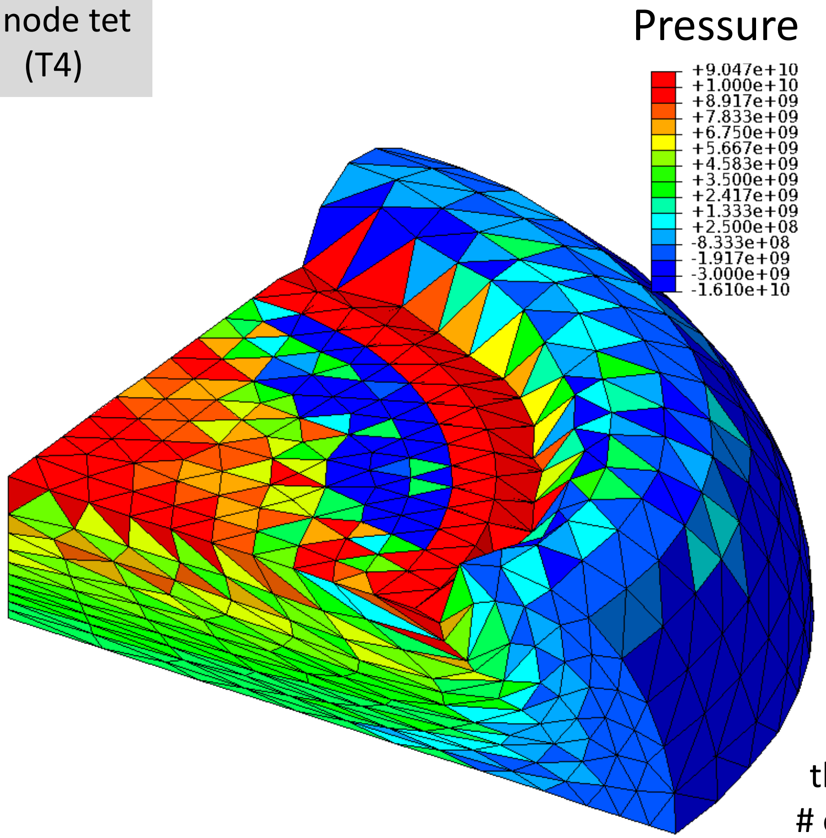
Issues in Conventional FE (ABAQUS)

e.g.) Barreling of Rubber Cylinder Neo-Hookean ***hyperelastic*** body with $\nu_{\mathrm{ini}} = \mathbf{0.49}$

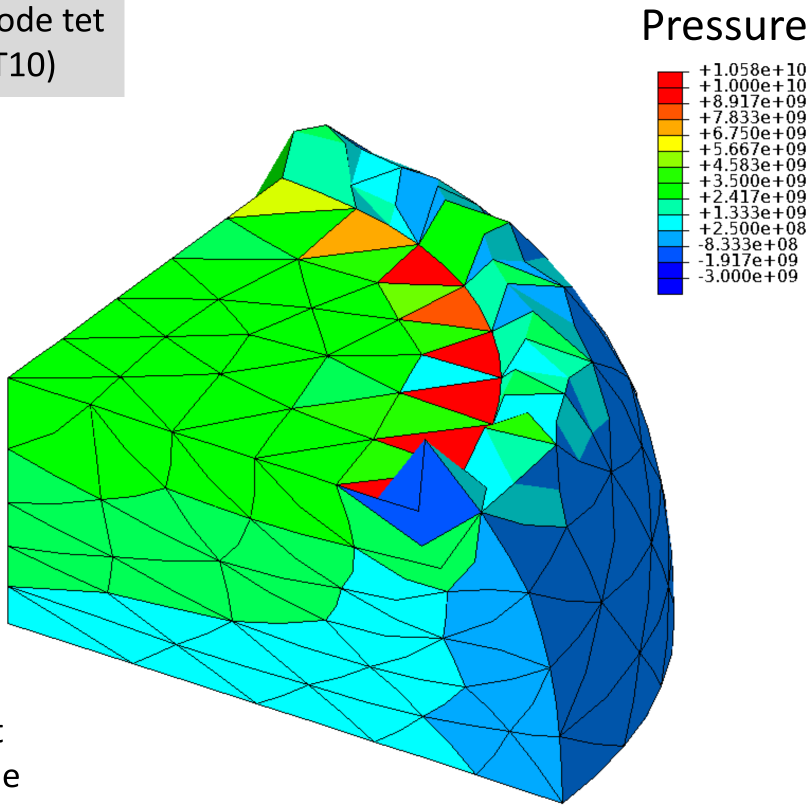
4 node tet (T4)

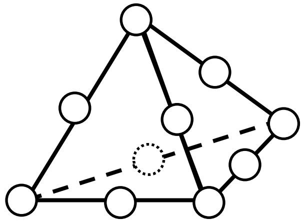


T4



10 node tet (T10)





T10

Almost the same # of nodes.

<u>ABQUS C3D4H</u>

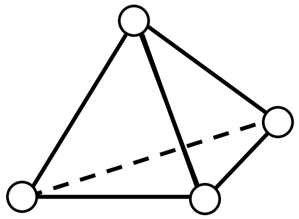
- ✓ No volumetric locking.
- ✗ Pressure checkerboarding.
- ✗ Shear locking & Corner locking.

<u>ABAQUS C3D10MH</u>

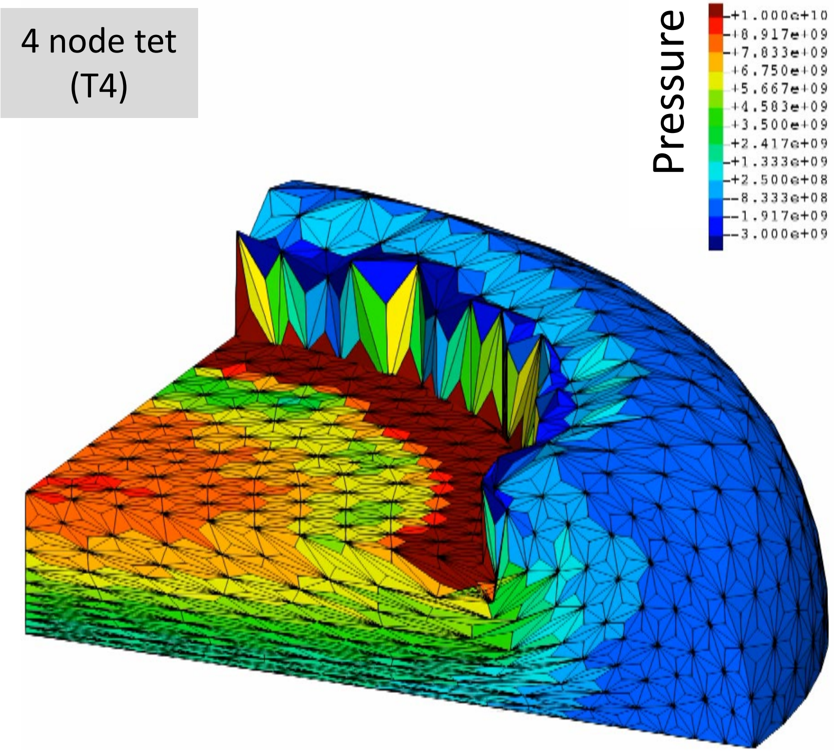
- ✓ No shear/volumetric locking.
- ✗ Short lasting (weak to severe deformation).
- ✗ Low interpolation accuracy.

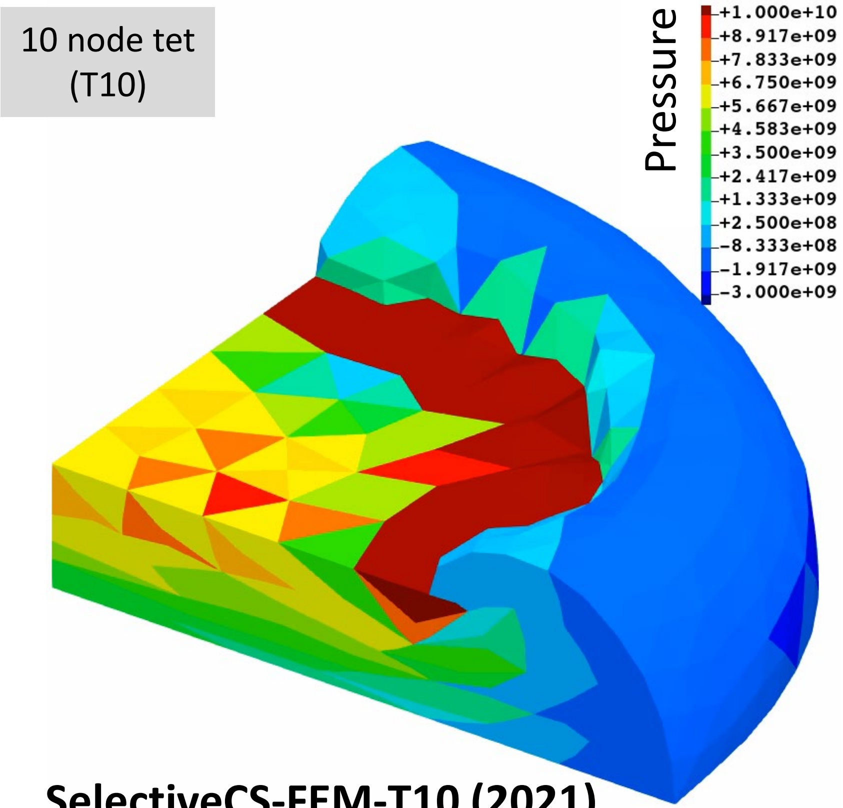
Our Approach using S-FEM

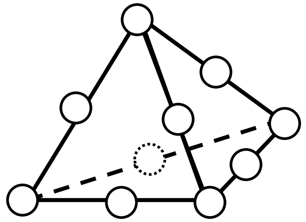
e.g.) Barreling of Rubber Cylinder Neo-Hookean ***hyperelastic*** body with $\nu_{\mathrm{ini}} = \mathbf{0.49}$



T4







T10

More than 10 times slower than FEM-T4, but no good idea for speed-up…

F-barES-FEM-T4 (2017)

- ✓ No shear/volumetric locking.
- ✓ Less pressure checkerboarding.
- ✓ Less corner locking. Long lasting.
- ✓ No oscillation in deviatoric stress.
- ✗ Long CPU time. Incompatible w/ FE.

SelectiveCS-FEM-T10 (2021)

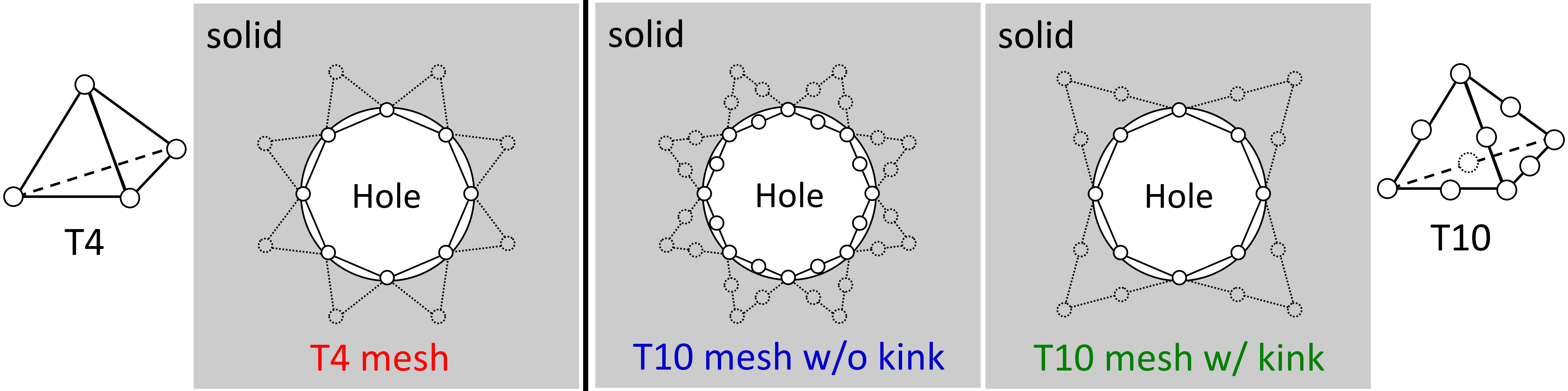
- ✓ No shear/volumetric locking.
- ✓ Less pressure checkerboarding.
- ✓ Less corner locking. Long lasting.
- ✗ Major oscillation in deviatoric stress.
- ✓ Same CPU time. Compatible w/ FE.

Cannot suppress stress oscillation, but no good idea for accuracy improvement…

Why not T10 but T4?

It is because T10 mesh is NOT good for the representation of complex geometries.

For example, surface mesh around a small hole looks like…



✓ is excellent with ES-FEM using minimal # of meshes.

✗ leads to a massive increase in DOF.

✗ leads to severe accuracy loss.

Also, the presence of mid-nodes leads to early convergence failure in large deformation.

Then, T4 is preferable for practical analyses with complex geometry.

Birth of a Next-gen S-FEM, EC-SSE, in 2022

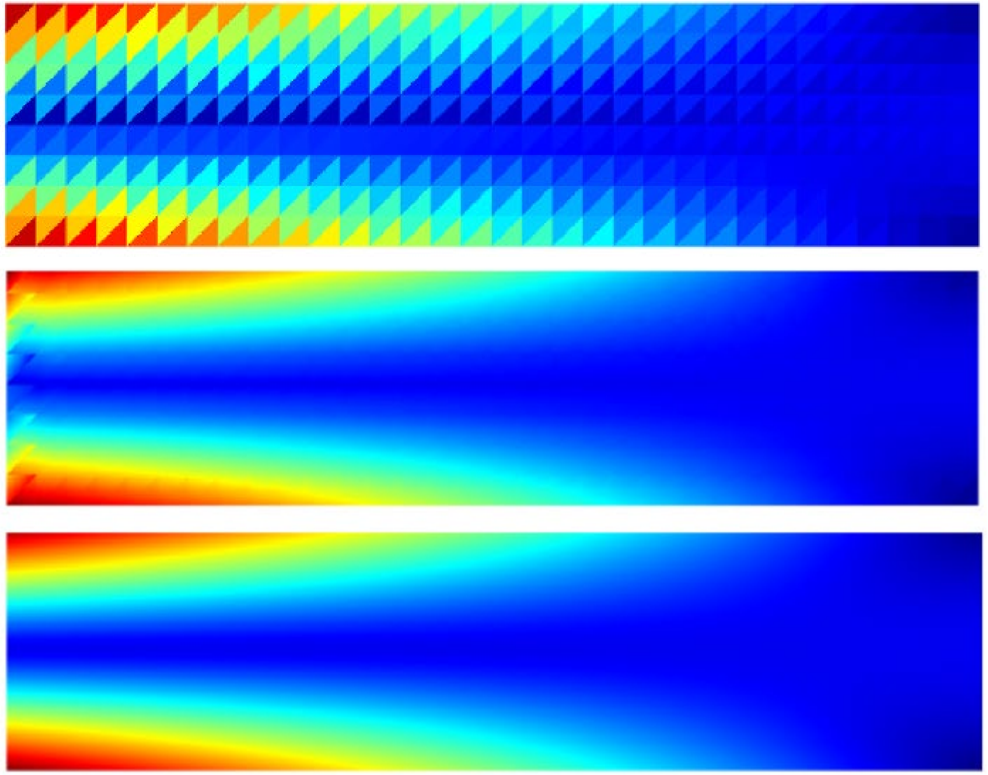
Comparison of Mises stress dist. in cantilever bending analyses in 2D

T. Jinsong *et al*., Euro. J. Mech. /A, v95, 2022.

FEM-T3

EC-SSE

Analytic



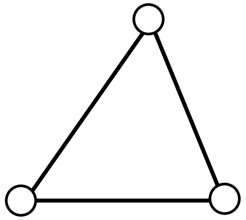
Edge
Center-based
Strain
Smoothing
Element

Detailed Later

- Step-like stress dist. (poor).
- Shear locking.

- Linear stress dist. (very good) using the same T3 mesh. Very close to analytical solution.
- No shear locking.

This breakthrough should be called "**S-FEM 2.0**".



T3

Poisson's Ratio: $\nu = 0.3$

EC-SSE is an excellent formulation for compressible solids;
but when $\nu \simeq 0.5$, EC-SSE has **volumetric locking** and **pressure checkerboarding**.
Therefore, EC-SSE is NOT directly applicable to nearly incompressible solids.

Objective

Develop a new S-FEM formulation to extend EC-SSE to nearly incompressible large deformation analysis

Strategy

Use the selective reduced integration (SRI)

- Use EC-SSE for the deviatoric part,
- Use NS-FEM for the volumetric part, and
- Combine them with SRI.

$\Bigg\}$ **EC-SSE-SRI**

Method

Introduction to ES-FEM, NS-FEM, EC-SSE, and EC-SSE-SRI

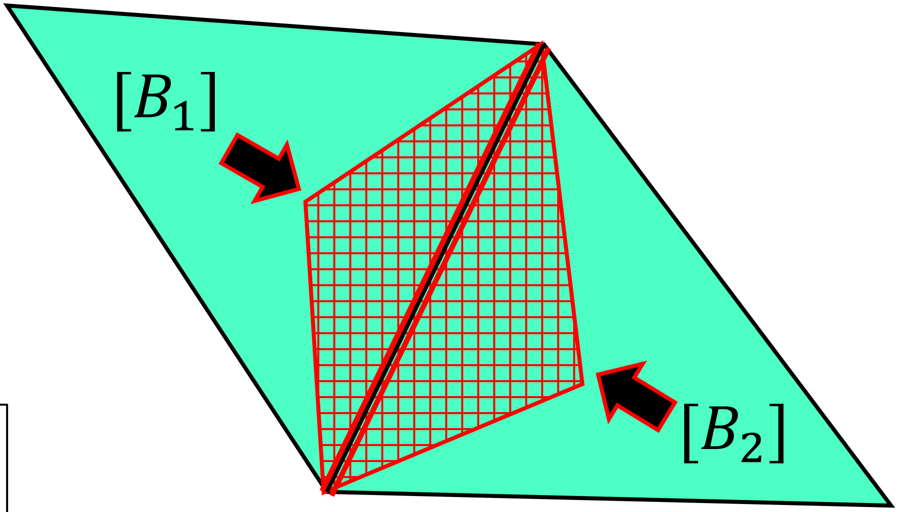
Brief of ES-FEM

Let us consider a mesh with only two 3-node triangular cells.

Let me explain in 2D for simplicity

- Make $[B](= \mathrm{d}\boldsymbol{N}/\mathrm{d}\boldsymbol{x})$ at each cell as usual.
- At each edge, gather $[B]$s of the connecting cells and average them with area weights to build $[{}^{\mathrm{Edge}}B]$.
- Calculate strain ($\boldsymbol{\varepsilon}$), stress ($\boldsymbol{\sigma}$) and nodal internal force $\{f^{\mathrm{int}}\}$ in each edge smoothing domain with $[{}^{\mathrm{Edge}}B]$.

As if putting a Gauss point on each edge center



Strain distribution is piecewise constant in each smoothing domain.

No shear locking.

Cannot avoid volumetric locking & pressure checkerboarding.

$[{}^{\mathrm{Edge}}B]$

↳ ${}^{\mathrm{Edge}}\boldsymbol{\varepsilon}$, ${}^{\mathrm{Edge}}\boldsymbol{\sigma}$, $\{{}^{\mathrm{Edge}}f^{\mathrm{int}}\}$ etc.

Brief of NS-FEM

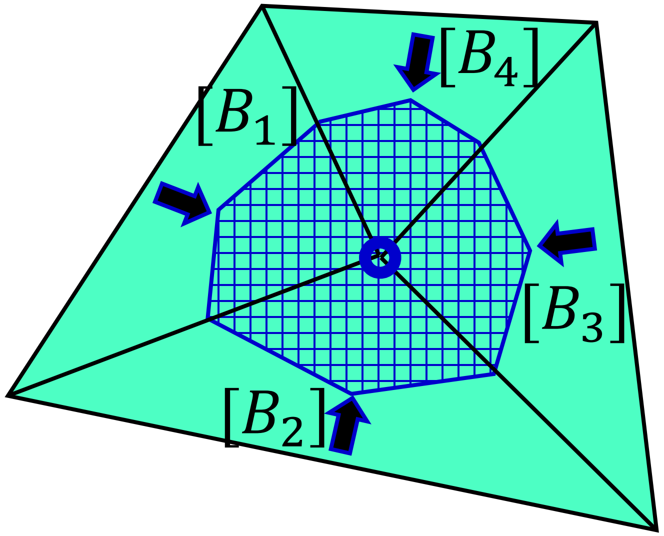
Let us consider a mesh with only four 3-node triangular cells.

Let me explain in 2D for simplicity

- Make $[B](= \mathrm{d}\boldsymbol{N}/\mathrm{d}\boldsymbol{x})$ at each cell as usual.
- At each node, gather $[B]$s of the connecting cells and average them with area weights to build $[{}^{\mathrm{Node}}B]$.
- Calculate strain ($\boldsymbol{\varepsilon}$), stress ($\boldsymbol{\sigma}$) and nodal internal force $\{f^{\mathrm{int}}\}$ in each nodal smoothing domain with $[{}^{\mathrm{Node}}B]$.

As if putting a Gauss point on each node center

Strain distribution is piecewise constant in each smoothing domain.



$[{}^{\mathrm{Node}}B]$

$\Rightarrow$ ${}^{\mathrm{Node}}\boldsymbol{\varepsilon}$, ${}^{\mathrm{Node}}\boldsymbol{\sigma}$, $\{{}^{\mathrm{Node}}f^{\mathrm{int}}\}$ etc.

No shar/volumetric locking.
Less pressure checkerboarding

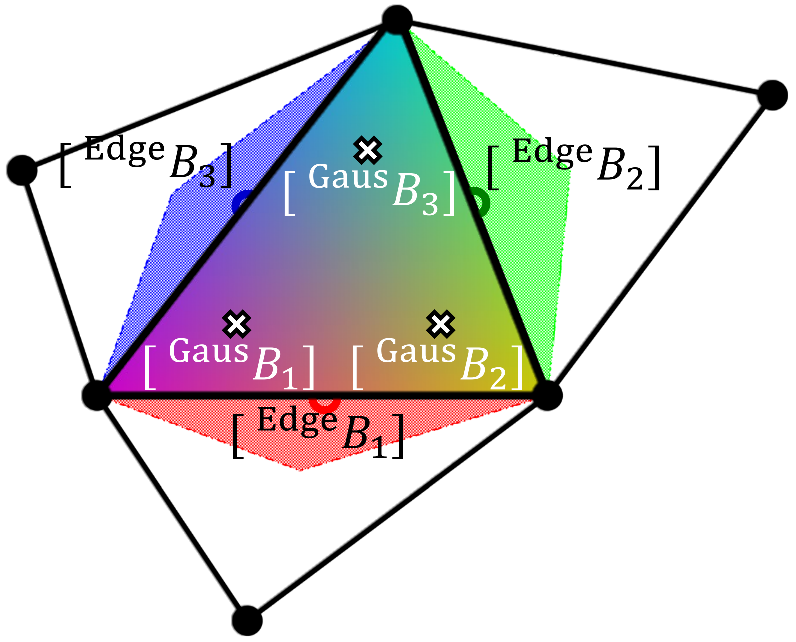
Cannot avoid spurious low-energy modes.

Brief of EC-SSE

- Make $[{}^{\mathrm{Edge}}B]$s in the same procedure as ES-FEM.
- Consider each $[{}^{\mathrm{Edge}}B]$ is the value at the center of each edge, and **assume $[\boldsymbol{B}]$ is linearly distributed in each cell**.
- Make three $[{}^{\mathrm{Gaus}}B]$s in each cell as the **extrapolation of the three $[{}^{\mathrm{Edge}}\boldsymbol{B}]$s**.
- Calculate ${}^{\mathrm{Gaus}}\varepsilon$, ${}^{\mathrm{Gaus}}\sigma$ and $\{f^{\mathrm{int}}\}$ using each $[{}^{\mathrm{Gaus}}B]$ in the same manner as the 2nd -order element.

Let me explain in 2D for simplicity

Conducting strain smoothing twice, the strain/stress are evaluated at each Gauss point.



Strain distribution is piecewise-linear in each cell and is **continuing at every edge center.**

- No shear locking with T3/T4 mesh.
- Fast mesh convergence rate in strain/stress as an 2nd –order element.

- Cannot avoid volumetric locking and pressure checkerboarding

Brief of EC-SSE-SRI (Our Latest Method)

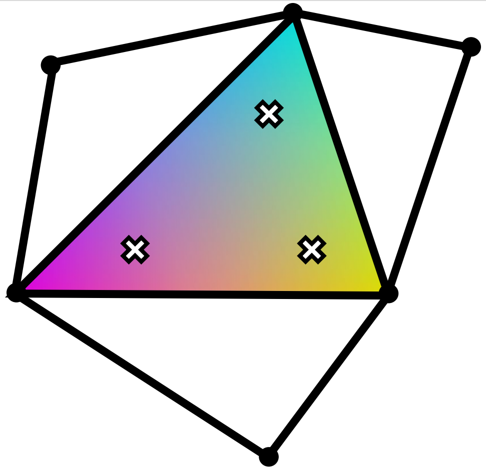
Apply the **selective reduced integration (SRI)** to EC-SSE to handle rubber-like solids

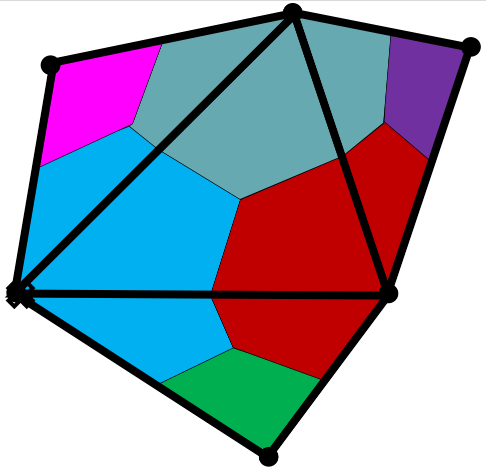
Let me explain in 2D for simplicity

(1) Calculate ε^{dev} at each Gauss point with EC-SSE

(3) Calculate ε^{vol} at each node with NS-FEM

Deviatoric Part





Volumetric Part

(2) Calculate σ^{dev} at each Gauss point and its contribution to $\{f^{\text{int}}\}$

(4) Calculate σ^{hyd} at each node and its contribution to $\{f^{\text{int}}\}$

Selective Reduced Integration (SRI)

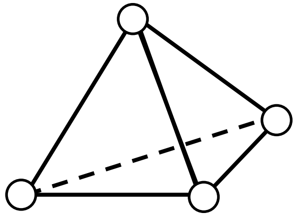
(5) Assemble $\{f^{\text{int}}\}$

Deviatoric strain distribution is <u>piecewise-linear in each cell</u> and is **<u>continuing at each edge center</u>.**

No shear/volumetric locking.
Less pressure checkerboarding.

Brief of EC-SSE-SRI-T4 (in 3D)

Let me explain with text only



[Deviatoric Part]

- Make $\left[{}^{\mathrm{Edge}}B\right]$ s in the same procedure as ES-FEM.
- **Make $\left[{}^{\mathrm{Face}}\boldsymbol{B}\right]$ s by re-smoothing three $\left[{}^{\mathrm{Edge}}\boldsymbol{B}\right]$s per face.**
- Consider each $[{}^{\mathrm{Face}}\boldsymbol{B}]$ is the value at the center of each **face**, and assume $[B]$ is linearly distributed in each cell.
- Make **four** $\left[{}^{\mathrm{Gaus}}\boldsymbol{B}\right]$ s in each cell as the extrapolation of the **four** $[{}^{\mathrm{Face}}\boldsymbol{B}]$ **s**.
- Calculate ${}^{\mathrm{Gaus}}\varepsilon_{\mathrm{dev}}$, ${}^{\mathrm{Gaus}}\sigma_{\mathrm{dev}}$ and $\{f_{\mathrm{dev}}^{\mathrm{int}}\}$ using each $[{}^{\mathrm{Gaus}}B]$, like the 2nd -order element.

[Volumetric Part]

- Make $\left[{}^{\mathrm{Node}}B\right]$s in the same procedure as NS-FEM.
- Calculate ${}^{\mathrm{Node}}\varepsilon_{\mathrm{vol}}$, ${}^{\mathrm{Node}}\sigma_{\mathrm{hyd}}$ and $\{f_{\mathrm{vol}}^{\mathrm{int}}\}$ using each $[{}^{\mathrm{Node}}B]$.

[SRI]

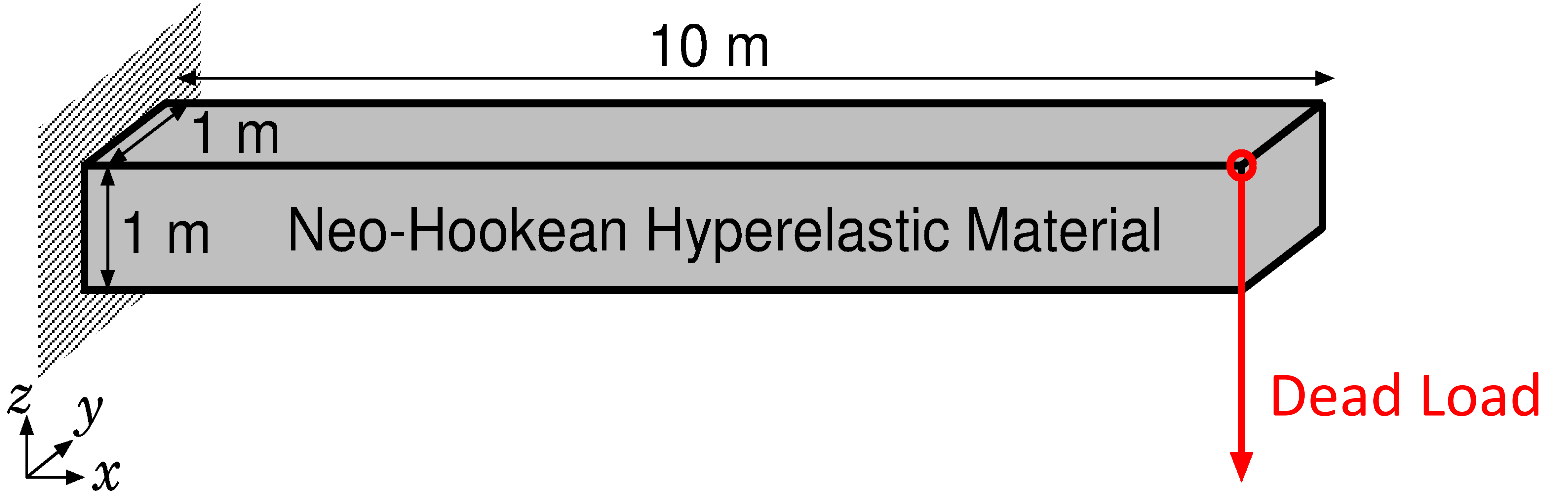
- Make $\{f^{\mathrm{int}}\} = \{f_{\mathrm{dev}}^{\mathrm{int}}\} + \{f_{\mathrm{vol}}^{\mathrm{int}}\}$.

Result & Discussion

Performance evaluation of EC-SSE-SRI-T4 in 3D and Discussion of CPU Cost

Bending of Rubber Cantilever

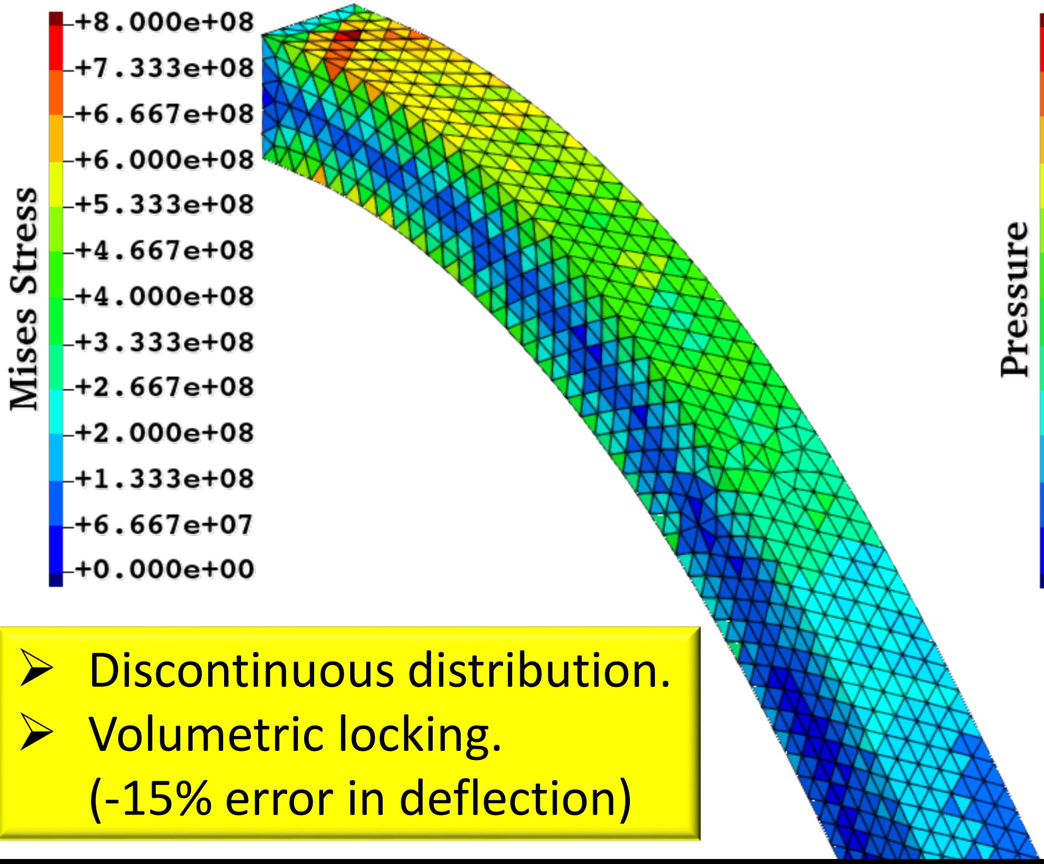
Outline



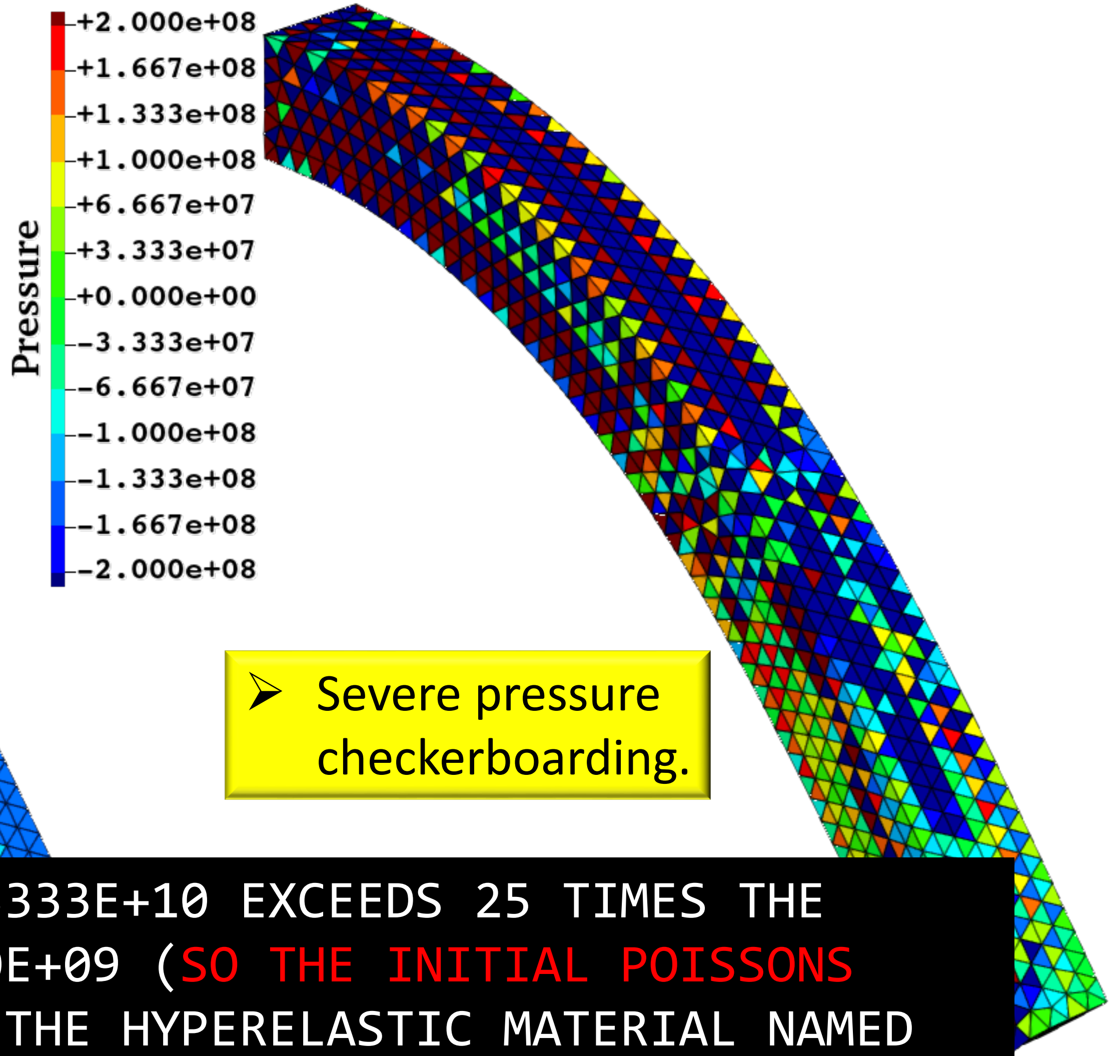
- 10 x 1 x 1 m cantilever.
- Neo-Hookean hyperelastic material, $E_{\mathrm{ini}} = 6\ \mathrm{GPa}$, $\boldsymbol{\nu}_{\mathbf{ini}} = \mathbf{0.49}$.
- Dead load applied to the tip node.
- A large deflection analysis with $u_z = -6.5\ \mathrm{m}$ at the final state.
- Compared the results of **ABAQUS C3D4** and EC-SSE-SRI-T4.

Bending of Rubber Cantilever

__Results of ABAQUS C3D4 (Final State)__



- ➢ Discontinuous distribution.
- ➢ Volumetric locking.
  (-15% error in deflection)



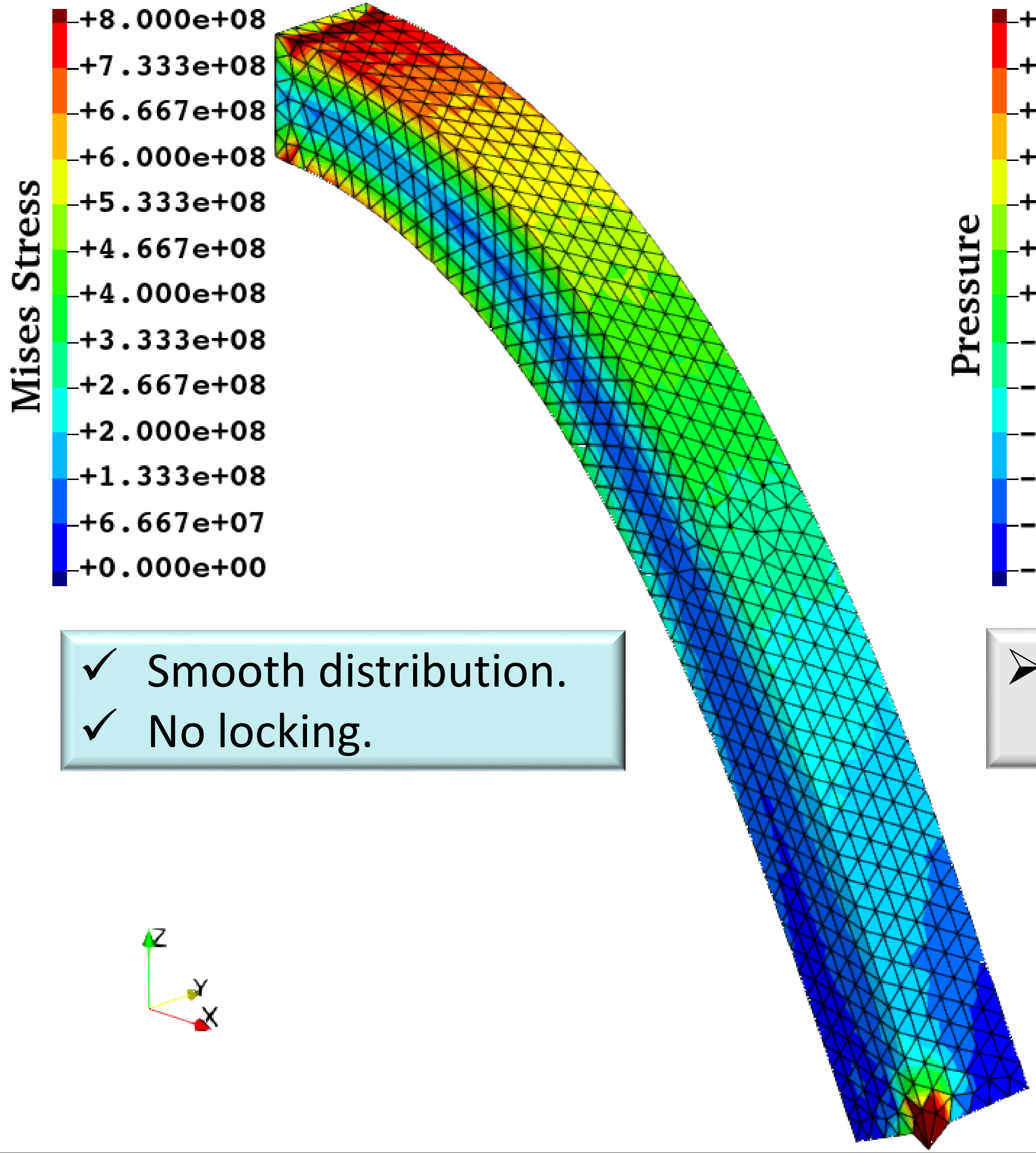
- ➢ Severe pressure checkerboarding.

ABAQUS dat file:

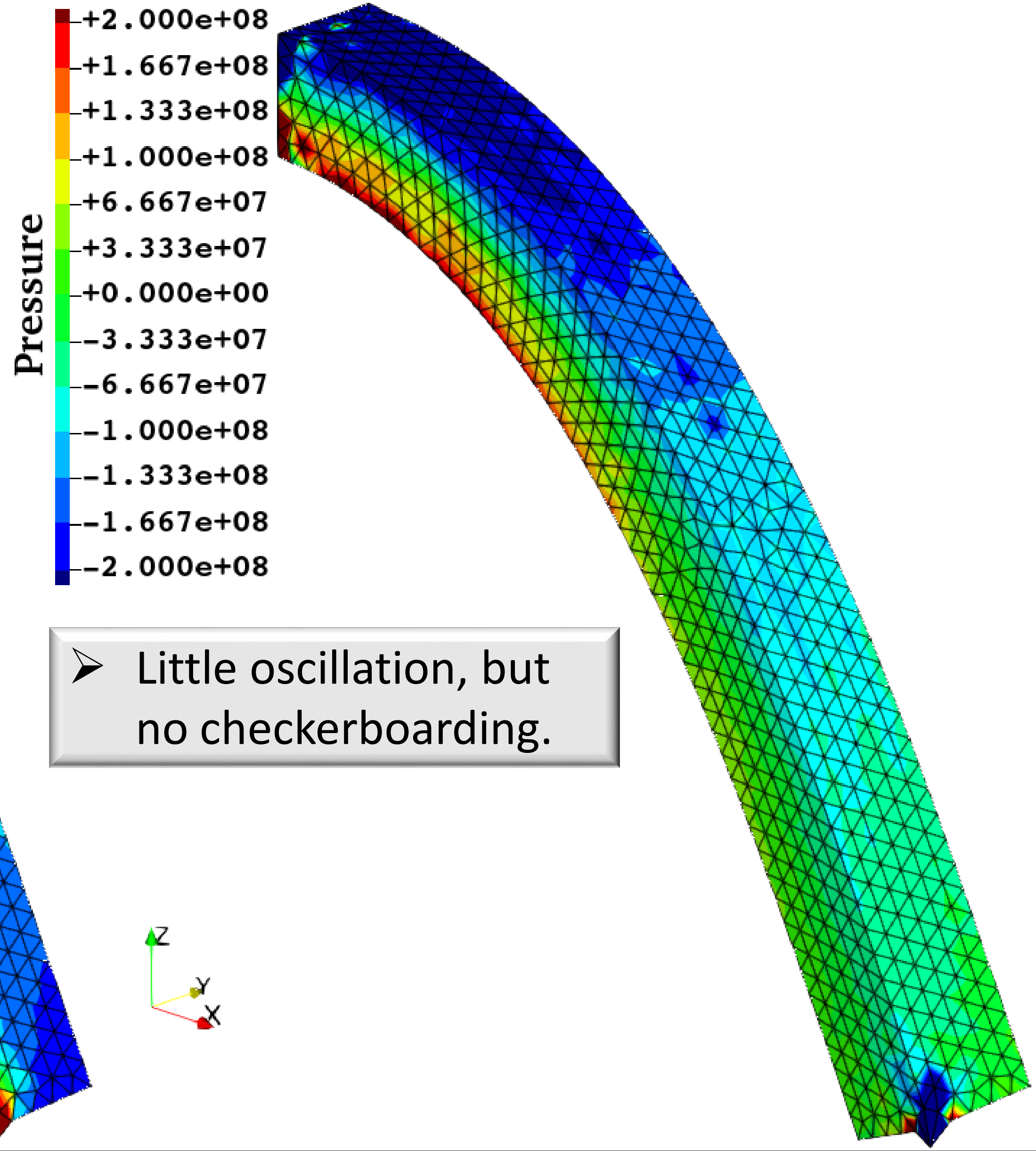
```
***WARNING: THE INITIAL BULK MODULUS OF 9.93333E+10 EXCEEDS 25 TIMES THE
            INITIAL SHEAR MODULUS OF 2.00000E+09 (SO THE INITIAL POISSONS
            RATIO 0.49000 EXCEEDS 0.48) FOR THE HYPERELASTIC MATERIAL NAMED
            MATERIAL-1. HOWEVER, A HYBRID TYPE ELEMENT IS NOT USED. THIS MAY
            CAUSE CONVERGENCE PROBLEMS.
```

Bending of Rubber Cantilever

Results of
EC-SSE
-SRI-T4
(Final State)



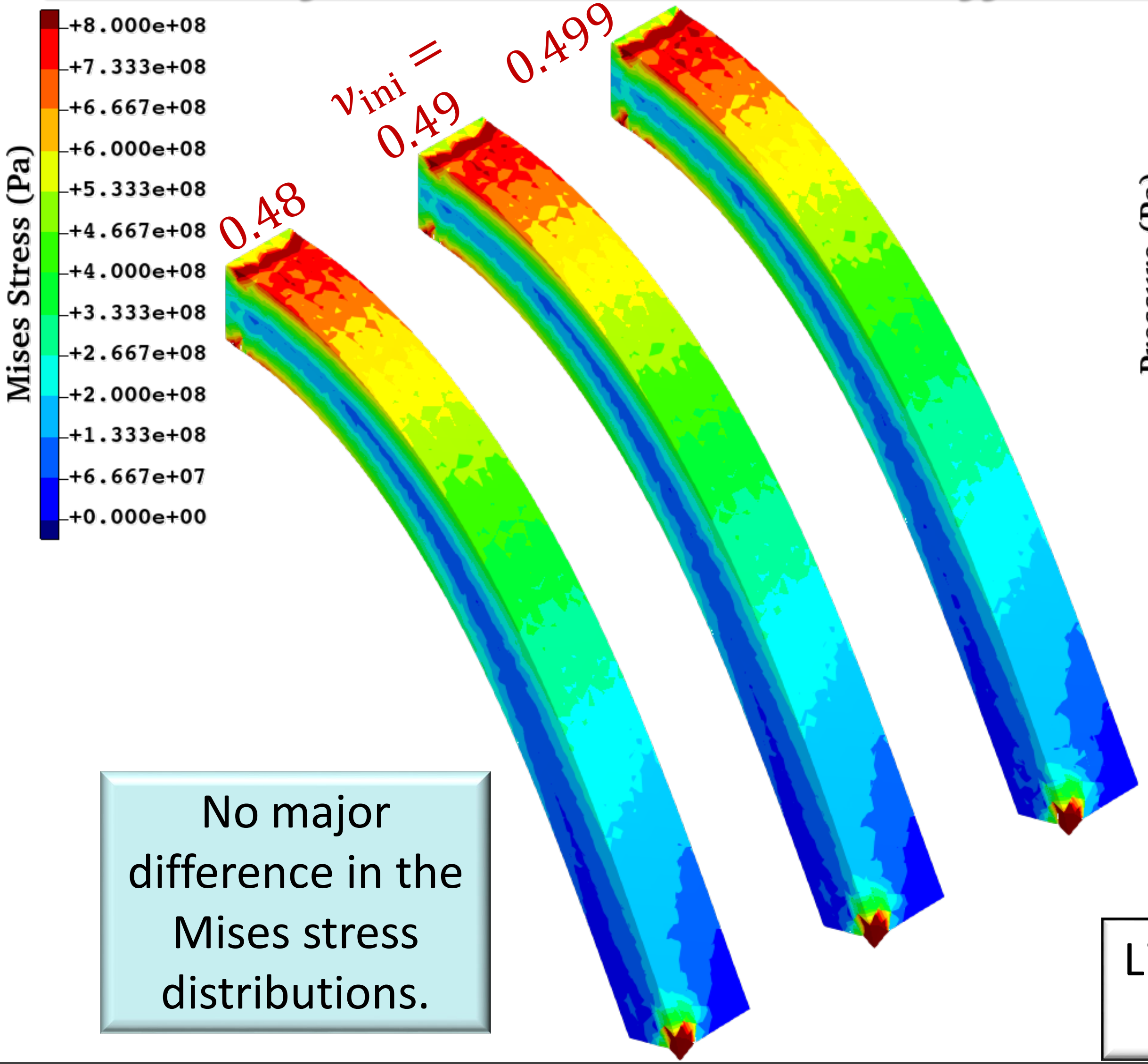
- ✓ Smooth distribution.
- ✓ No locking.

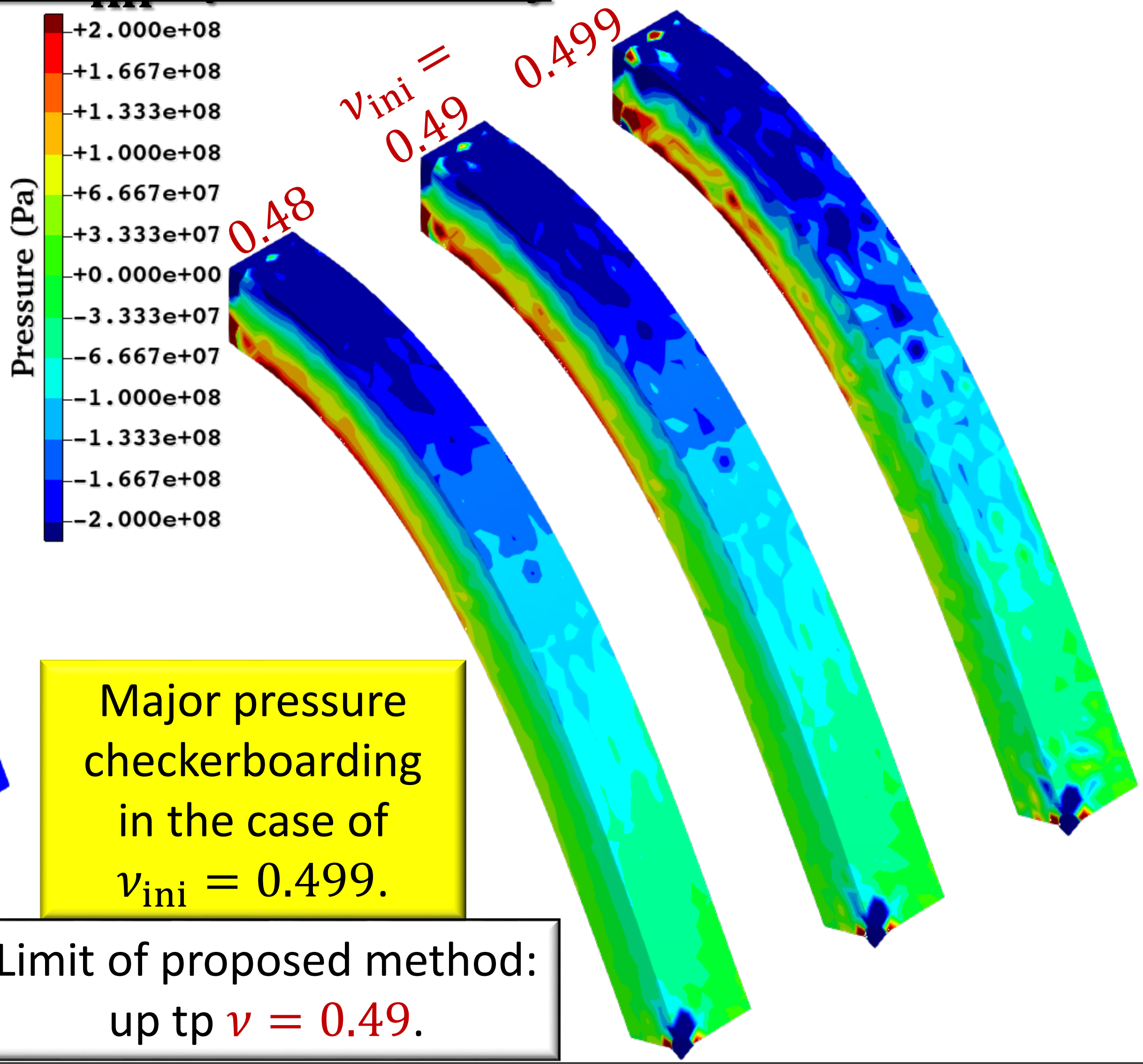


- ➢ Little oscillation, but no checkerboarding.

Bending of Rubber Cantilever

Results of EC-SSE-SRI-T4 with Different ν_{ini}s (Final State)





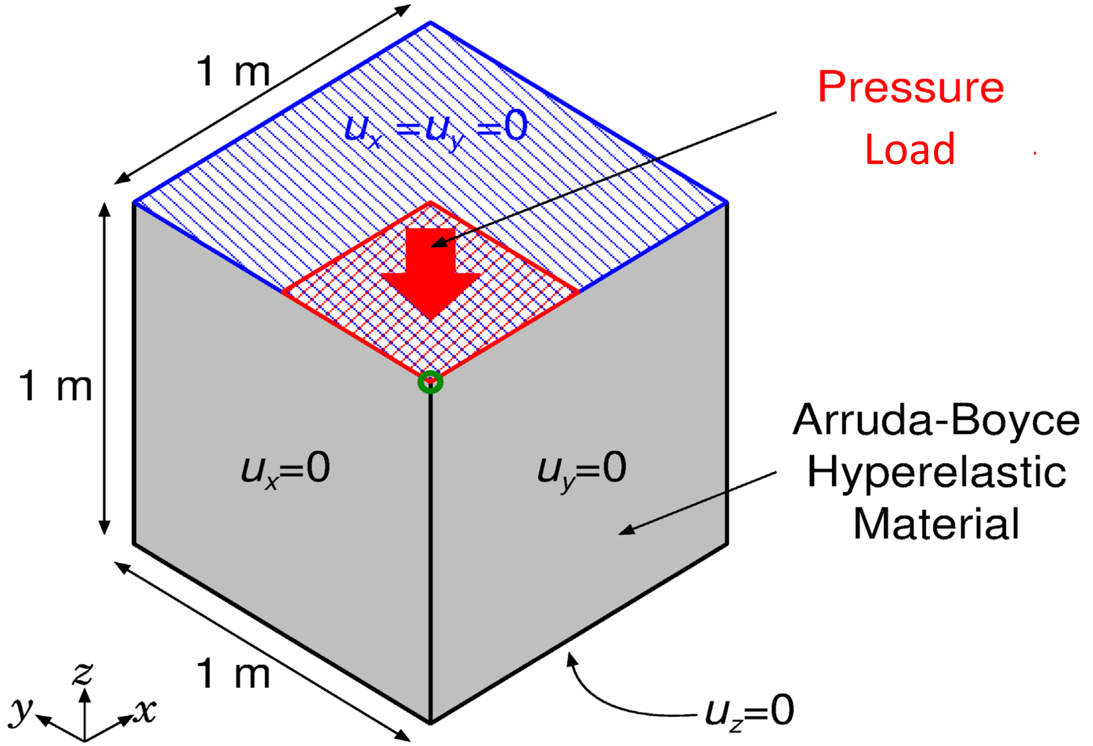
No major difference in the Mises stress distributions.

Major pressure checkerboarding in the case of $\nu_{\mathrm{ini}} = 0.499$.

Limit of proposed method: up tp $\nu = 0.49$.

Pressuring of Rubber Block

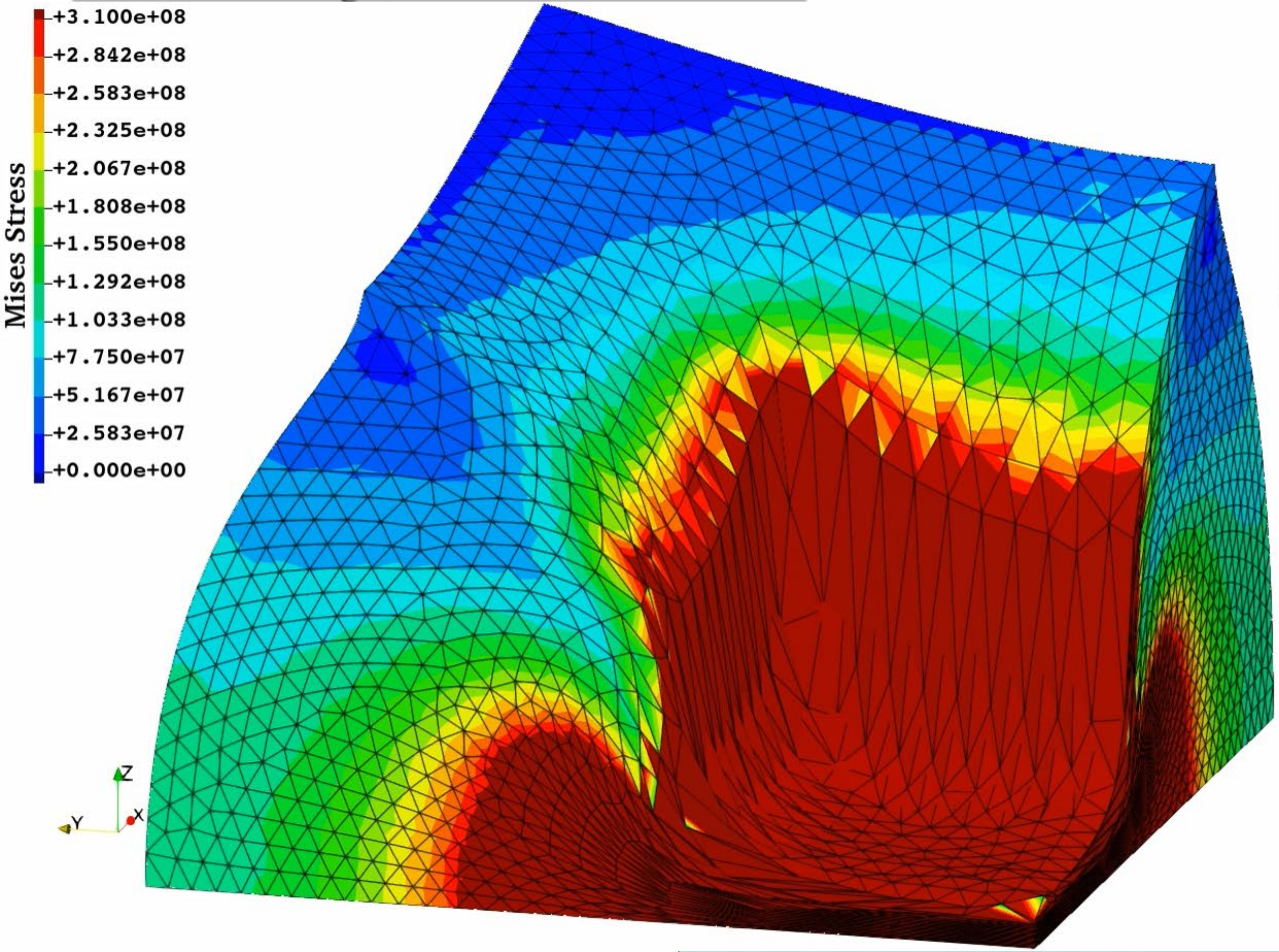
Outline

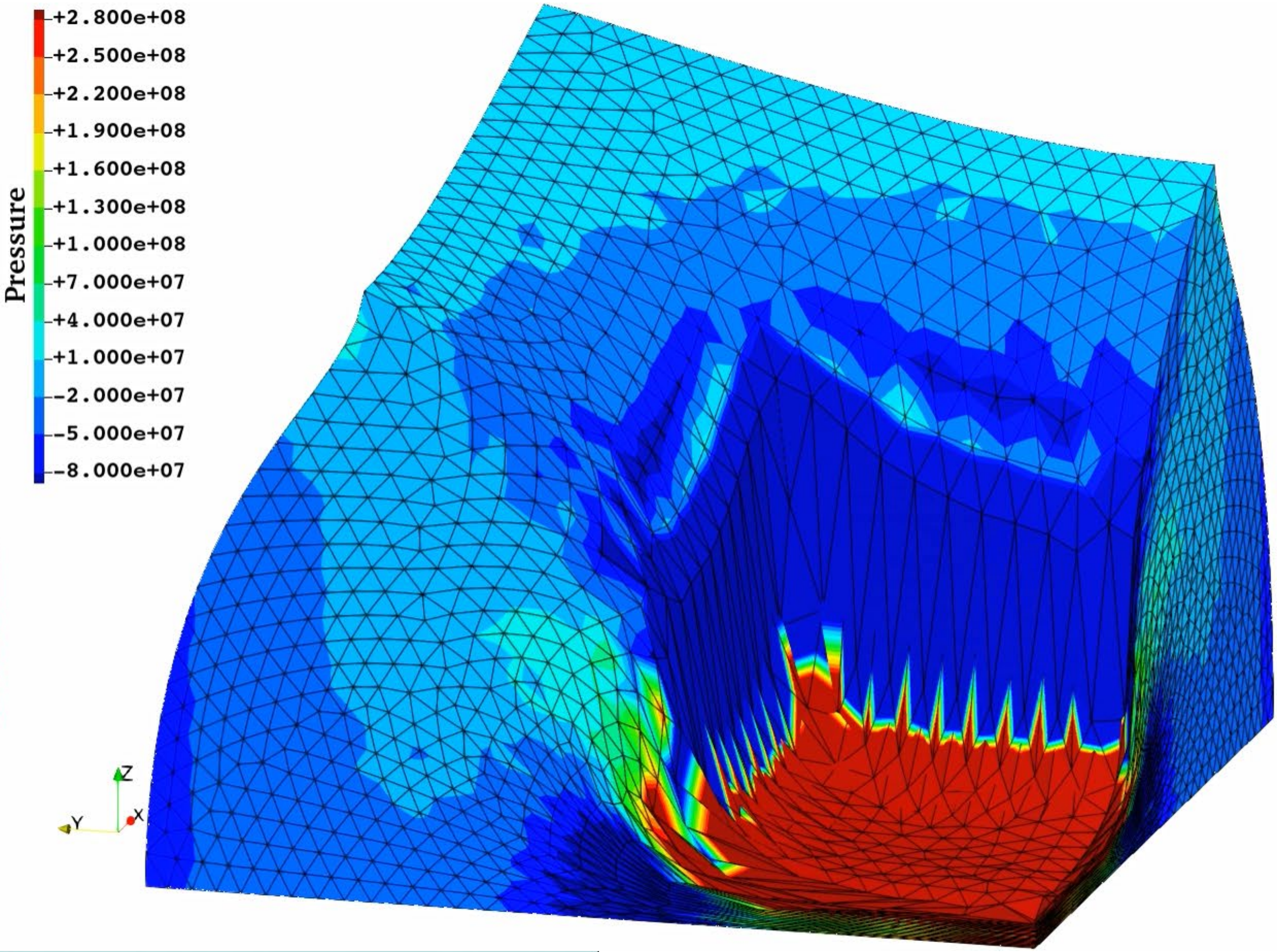


- 1 x 1 x 1 m block.
- Arruda-Boyce hyperelastic material, $E_{\mathrm{ini}} = 24$ GPa, $\boldsymbol{\nu_{\mathrm{ini}} = 0.49}$.
- Applying pressure on ¼ of the top face with lateral confinement.
- Evaluated the result of EC-SSE-SRI-T4.

Pressuring of Rubber Block

Results of EC-SSE-SRI-T4

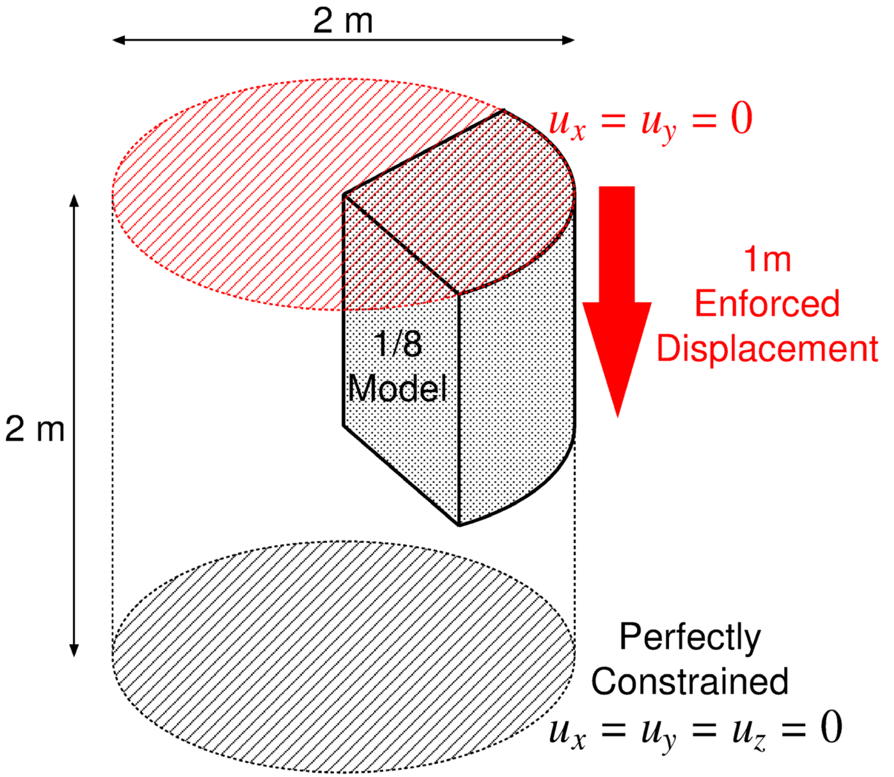




- ✓ No big issue in stress distributions
- ✓ Sufficient large deformation robustness

Barreling of Rubber Cylinder

Outline



- 1 m cylinder in radius and height.
- Neo-Hookean hyperelastic material, $E_{\mathrm{ini}} = 6$ GPa, $\boldsymbol{\nu}_{\mathbf{ini}} = \mathbf{0.49}$.
- Applying enforced compression displacement on the top face with lateral confinement.
- Evaluated the result of EC-SSE-SRI-T4.

Static Implicit

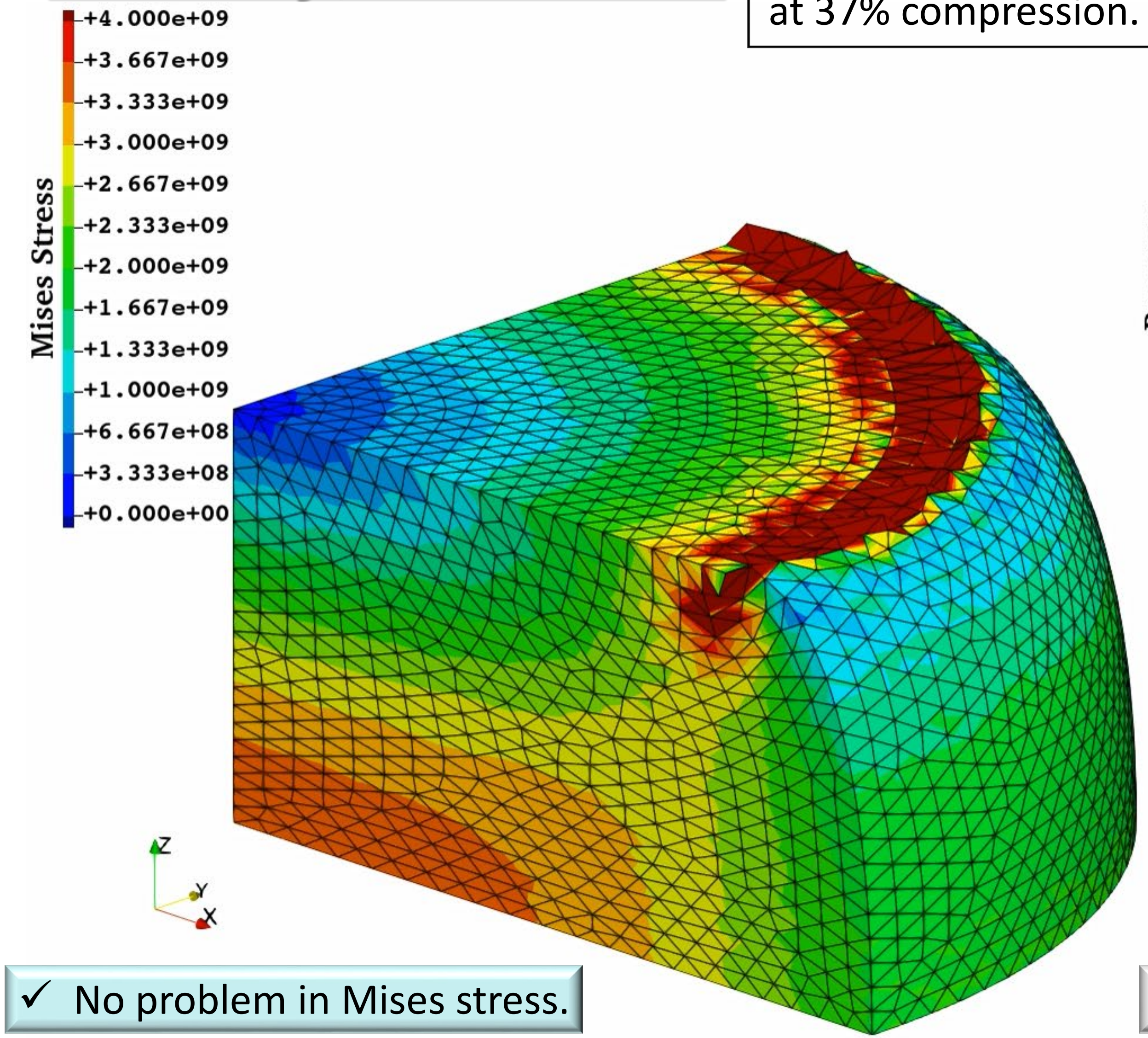
Barreling of Rubber Cylinder

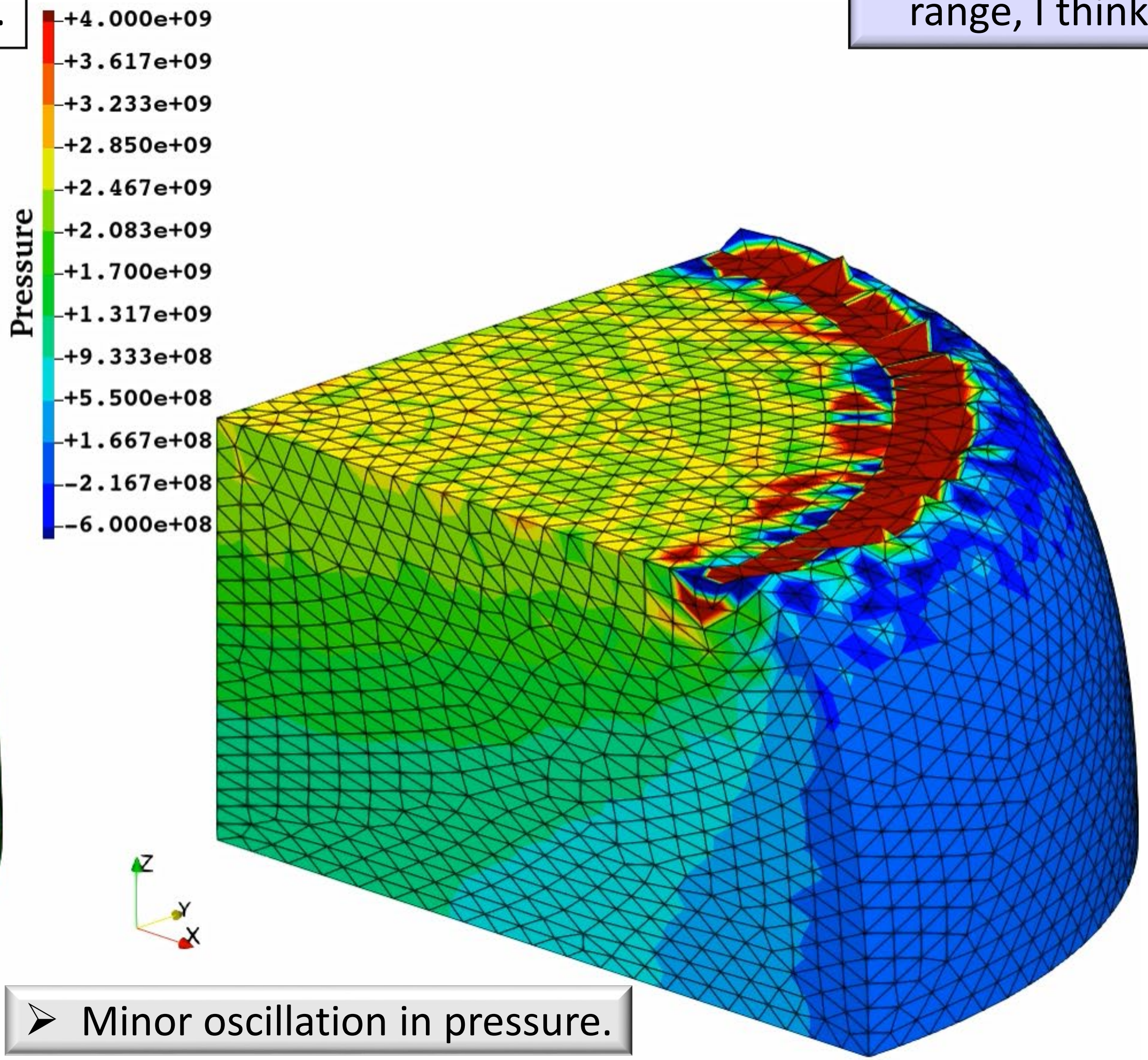
Results of EC-SSE-SRI-T4

Convergence failure at 37% compression.

∴ acceptably robust in large deformation

Within acceptable range, I think.



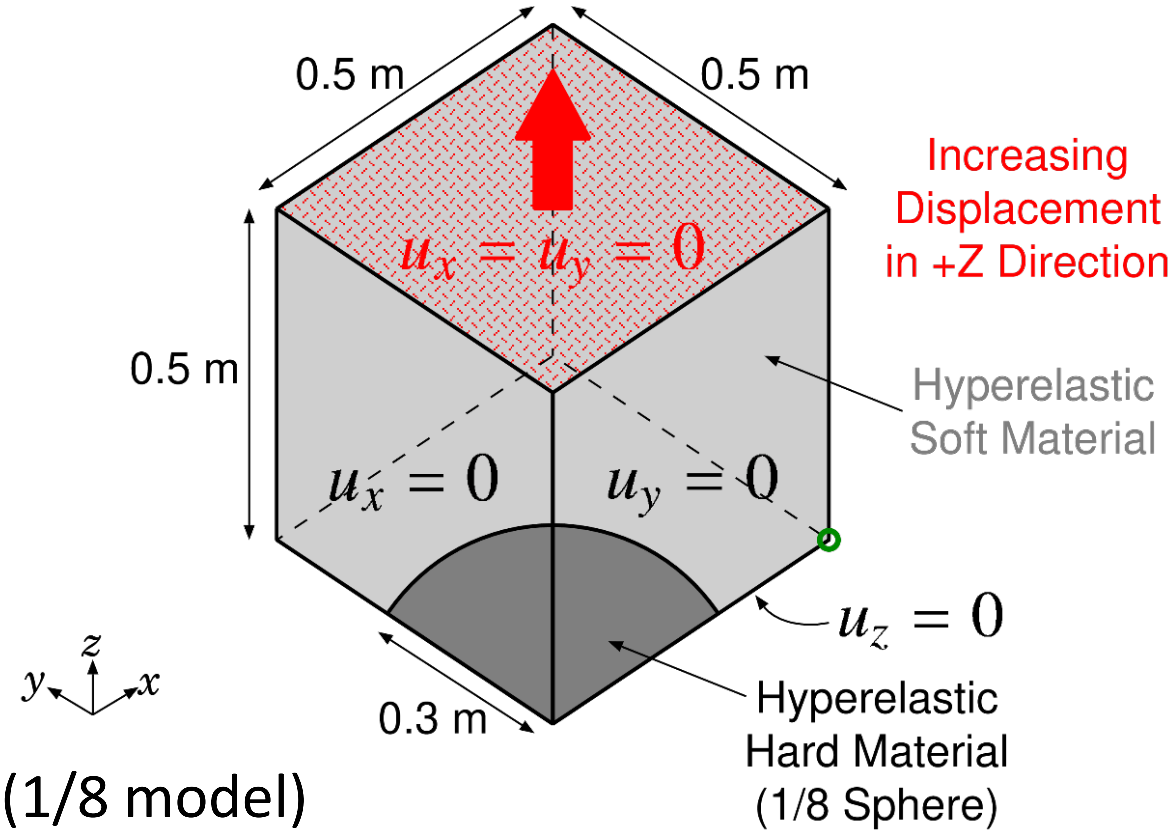


✓ No problem in Mises stress.

➢ Minor oscillation in pressure.

Tensioning of Rubber-Filler Composite

Outline



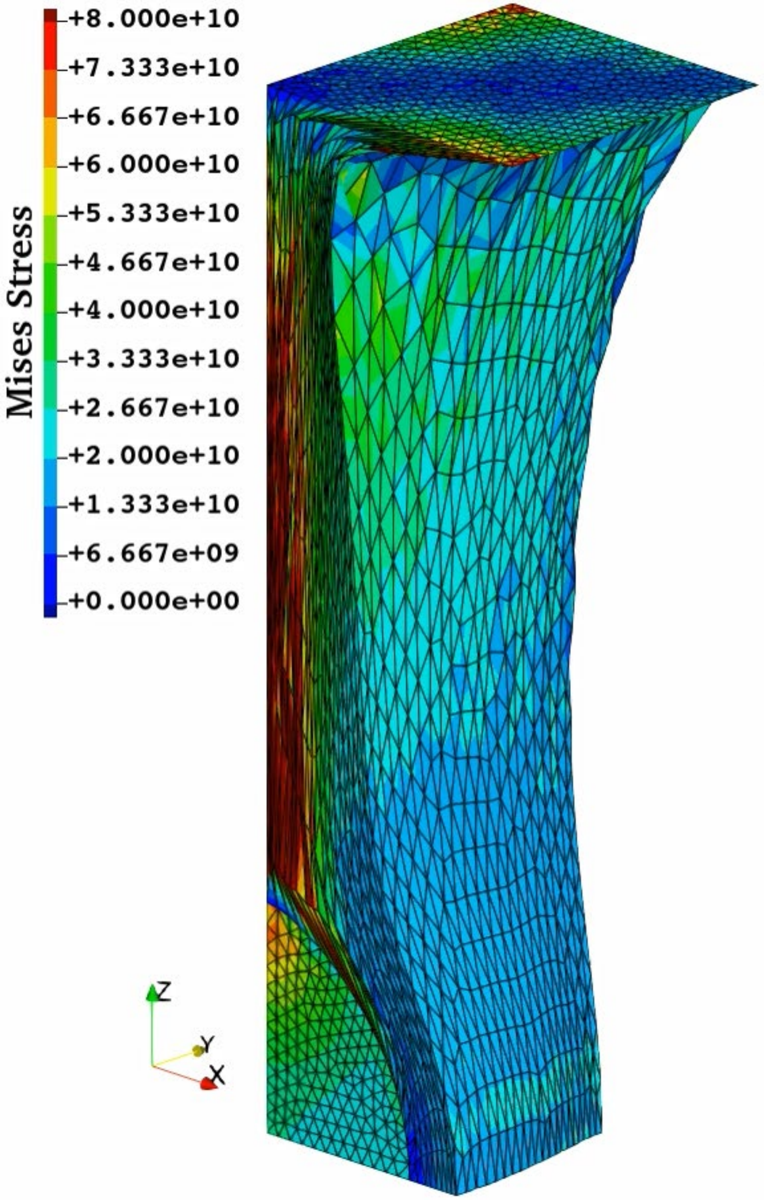
- 0.5 x 0.5 x 0.5 m cube (1/8 model)
- Rubber: Neo-Hookean hyperelastic material ($E_{\mathrm{ini}} = 6$ GPa, $\boldsymbol{\nu_{\mathrm{ini}} = 0.49}$)
- Iron Filler: Neo-Hookean hyperelastic material ($E_{\mathrm{ini}} = 260$ GPa, $\boldsymbol{\nu_{\mathrm{ini}} = 0.3}$)
- Applying enforced tensioning displacement on the top face with lateral confinement.
- Evaluated the result of EC-SSE-SRI-T4.

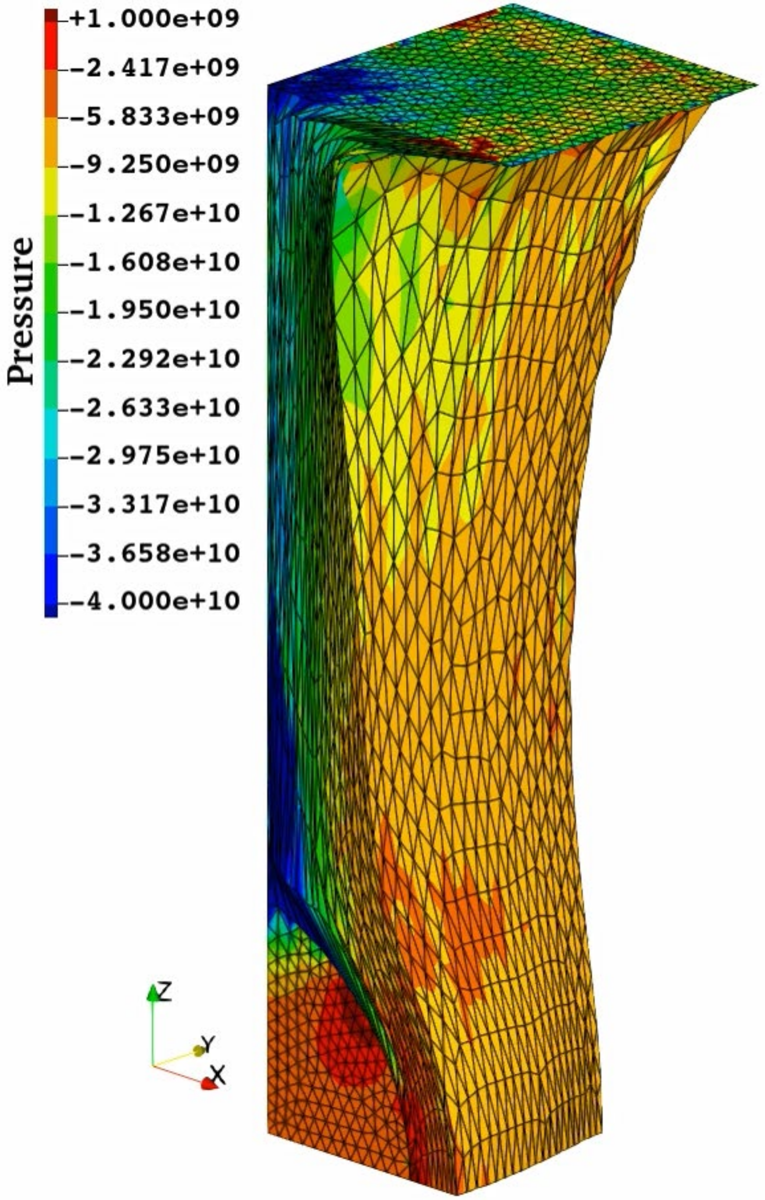
Tensioning of Rubber-Filler Composite

Results of
EC-SSE
-SRI-T4

Convergence failure at 221% stretch
∴ sufficiently robust in large deformation

✓ No issue in Mises stress.





➢ Minor pressure oscillation only in rubber part.

Within acceptable range, I think.

Discussion on CPU Time of EC-SSE-SRI-T4

- Since the most of CPU time for implicit analyses is spent solving the stiffness equation (i.e., $[K]\{u\} = \{f\}$), the size of $[K]$ matrix (N) directly affects the CPU time.
- EC-SSE-SRI-T4 is a purely displacement-based FE formulation; thus, the matrix size (N) is exactly identical to that of FEM-T4.
- EC-SSE-SRI-T4 conducts strain smoothing across FE cells; thus, the matrix bandwidth of $[K]$ is x6.7 wider than that of FEM-T4.

Formulation	Bandwidth of $[K]$	v.s. FEM-T4 Ratio
FEM-T4	14 nodes x 3 DOF	1
FEM-T10	28 nodes x 3 DOF	2.0
ES-FEM-T4	45 nodes x 3 DOF	3.2
NS-FEM-T4, SelectiveES/NS-FEM	60 nodes x 3 DOF	4.3
EC-SSE-T4, EC-SSE-SRI-T4	94 nodes x 3 DOF	6.7

- Therefore, as for calculation speed, EC-SSE-SRI-T4 is about x6.7 slower than FEM-T4.

Discussion on CPU Time of EC-SSE-SRI-T4

- Meanwhile, we should remind that
  - FEM-T4 cannot avoid volumetric locking and pressure checkerboarding,
  - FEM-T10 cannot have large deformation robustness (short-lasting),

  no matter how fine the mesh is.
- Therefore, I believe, EC-SSE-SRI-T4 **is practically acceptable and worth using**, even though the CPU time is 7 times longer than FEM-T4.

What do you think?

Summary

Summary

- A next-gen S-FEM, EC-SSE-SRI-T4, was proposed to handle large deformation of rubber-like solids.
- The performance of EC-SSE-SRI-T4 is summarized as follows:
  - **No shear/volumetric locking.**
  - **Minor pressure checkerboarding** when $\boldsymbol{\nu} \leq \mathbf{0.49}$.
  - **7 times longer CPU time** than FEM-T4 when using the same mesh.
  - ➢ More accurate than conventional T4 elements and SelectiveES/NS-FEM-T4.
  - ➢ More robust (long-lasting) than conventional T10 elements.
- The EC-SSE family would be the standard T4 formulation in the near future.

Thank you for your kind attention!

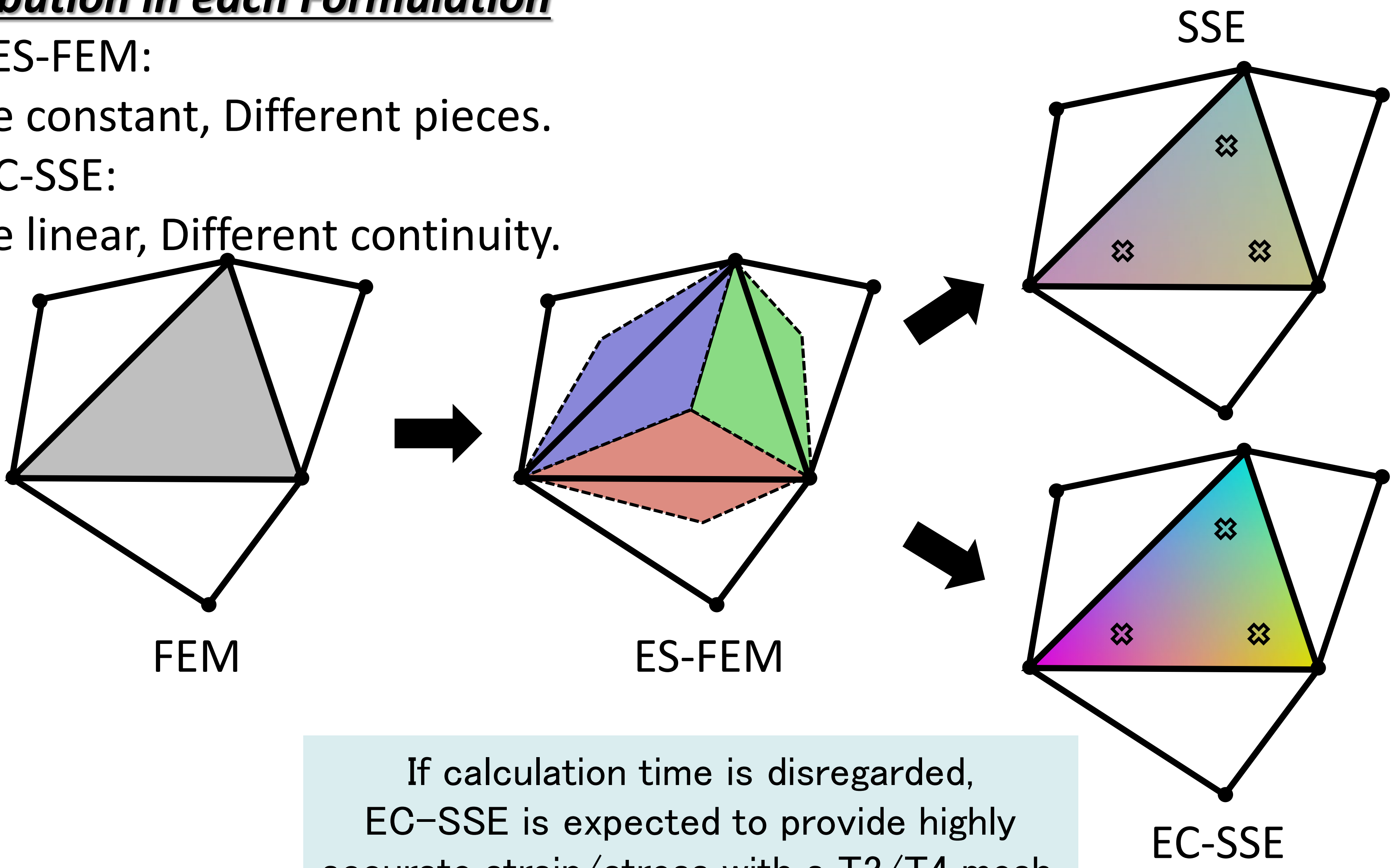
Appendix

Difference in Formulation

Strain Distribution in each Formulation

- FEM and ES-FEM:
  Piece-wise constant, Different pieces.
- SSE and EC-SSE:
  Piece-wise linear, Different continuity.

Let me explain in 2D for simplicity



If calculation time is disregarded, EC−SSE is expected to provide highly accurate strain/stress with a T3/T4 mesh.

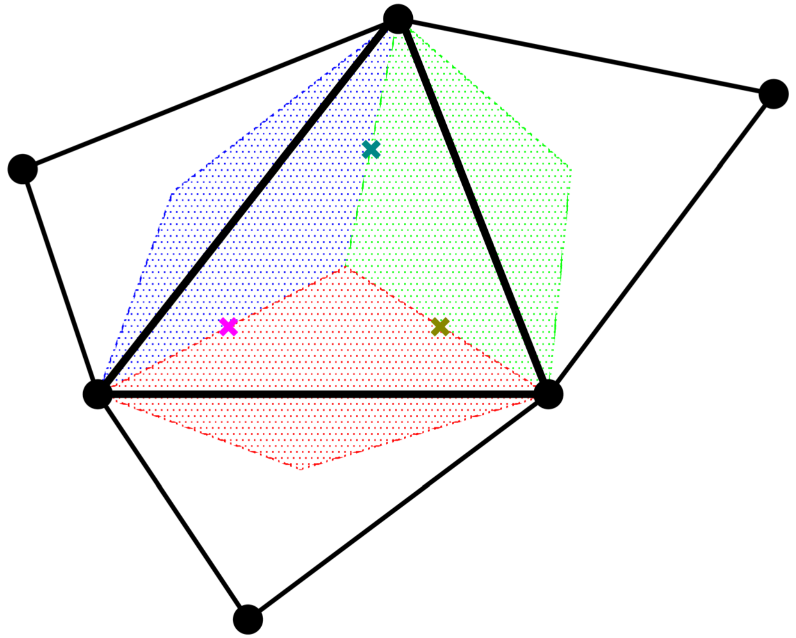
Brief of SSE

- Make [${}^{\text{Edge}}B$]s in the same procedure as ES-FEM.
- Consider each [${}^{\text{Edge}}B$] represents $[B]$ in its edge smoothing domain, and **assume $[B]$ is linearly distributing in each cell**.
- Make three [${}^{\text{Gaus}}B$]s in each cell as the **average of neighbor two [${}^{\text{Edge}}B$]s** .
- Calculate ${}^{\text{Gaus}}\varepsilon$, ${}^{\text{Gaus}}\sigma$ and $\{f^{\text{int}}\}$ using each $[{}^{\text{Gaus}}B]$ in the same manner as the 2nd -order element.

Let me explain in 2D for simplicity

Conducting strain smoothing twice, the strain/stress are evaluated at each Gauss point.

Strain distribution is piecewise-linear in each cell



- No shear locking with T3/T4 mesh.
- Fast mesh convergence rate in strain/stress.

- Cannot avoid volumetric locking and pressure checkerboarding